ORDER STATISTICS AND OUTLIERS

DISSERTATION

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BY

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PREFACE

In a set of data, outlier is an observation (or set of observations) which appears to be inconsistent with the remainder of set of data. Order statistics has immense role in outliers problem. In the reference of order statistics, outliers are those order statistics which have different distribution from other order statistics. Non identically distributed random variables in any sample, are examples of outliers.

In this dissertation an attempt has been made to present the available up to date literature on recurrence relations for outlier models. The whole dissertation is divided into five chapters.

Chapter-I deals with the basic concepts and results needed in the subsequent chapters.

In chapter-II, recurrence relations for single outlier model is given.

Chapter-III deals with the recurrence relations for independent non identically distributed random variables.

Chapter-IV, embodies recurrence relations for nonindependent and nonidentically distributed random variables.

Chapter-V deals recurrence relations for two related symmetric outlier models.

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Department of Statistics & Operations Research

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Chapter-I

PRELIMINARIES

1.1 ORDER STATISTICS

If the random variables $X_1, X_2, \ldots, X_n$ are arranged in the ascending order of magnitude $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ then $X_{r:n}$ or $X_{(r)}$ is called the r^{th} order statistics in a sample of size n . The two terms $X_{1:n} = \min (X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max (X_1, X_2, \ldots, X_n)$ are called extremes.

The subject of order statistics deals with the properties and applications of these ordered random variables and of functions involving them (David, 1981). It is different from the rank order statistics in which the order of value of observation rather than its magnitude is considered.

It plays an important role both in model building and in statistical inference. For example extreme(largest, smallest) values are important in oceanography(waves and tides), material sterength(strength of a chain depends on the weakest link) and meteorology (extremes of temperature, pressure etc.).

Order Statistics have immense application in life testing and relaibility problem. If n similar items are simultaniously placed on life test, the life of the first item to fail is first order statistics, life of the second item to fail is second order statistics and so on. Often experimenter may wish to terminate the experiment when only m ($< n$) failures have occured to save the resources and time. In this case we have only the first m order statistics on the basis of which we have to make inferences.

In statistical inference $X_{n:n} - X_{1:n}$ (range) is widely used to estimate the standard deviation (David, 1981). It is also used in outlier's detection (Barnett, 1984).

For further applications, one may refer to Malik *et al.* (1988), Harter(1978), Gumbel(1958) and Galambos(1978).

1.2 PROBABLITY DENSITY FUNCTION AND DISTRIBUTION FUNCTION OF ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous pdf. $f(x)$ and cdf $F(X)$. Then the pdf of $X_{r:n}$ $1 \le r \le n$, the r^{th} order statistics, is given by

$$f_{r:n}(x) = C_{r:n} \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r} f(x) \quad , \quad -\infty < x < \infty \qquad \ldots(1.2.1)$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} = \left[\beta(r, n-r+1) \right]^{-1} = \frac{n!}{(r-1)!(n-r)!} \qquad \ldots(1.2.2)$$

and cdf is

$$F_{r:n}(x) = P (X_{r:n} \le x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} \left[F(x) \right]^{i} \left[1 - F(x) \right]^{n-i} \qquad \ldots(1.2.3)$$

$$F_{r:n}(x) = C_{r:n} \int_0^{F(x)} t^{r-1} (1 - t)^{n-r} dt \qquad \ldots(1.2.4)$$

where (1.2.4) is incomplete beta function for X continuous, (1.2.1) can be obtained from (1.2.4) by differentiating w.r.t. x.

In particular

$$F_{1:n}(x) = 1 - (1 - F(x))^{n} \qquad ...(1.2.5)$$

$$F_{n:n}(x) = (F(x))^{n} \qquad ...(1.2.6)$$

The joint pdf. of $X_{r:n}$ and $X_{s:n}$ ($1 \le r < s \le n$) is given by

$$f_{r,s:n}(x,y) = C_{r,s:n} \Big[F(x) \Big]^{r-1} \Big[F(y) - F(x) \Big]^{s-r-1} \Big[1 - F(y) \Big]^{n-s} f(x)\ f(y) \quad , \quad -\infty < x < y < \infty \qquad ...(1.2.7)$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!\ (s-r-1)!\ (n-s)!} = \frac{1}{\beta(r,\ s-r,\ n-s+1)} \qquad ...(1.2.8)$$

$$F_{r,s:n}(x,y) = P\ (X_{r:n} \le x,\ X_{s:n} \le y\ )$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!\ (j-i)!\ (n-j)!} \Big[F(x) \Big]^{i} \Big[F(y) - F(x) \Big]^{j-i} \Big[1 - F(y) \Big]^{n-j} \qquad (1.2.9)$$

1.3 SINGLE AND PRODUCT MOMENTS OF ORDER STATISTICS

Let $\nu_{r:n}^{(k)}$ be the k^{th} moment of the r^{th} order statistics and $\nu_{r,s:n}^{(j,k)}$ be the product moment of the j^{th} power of r^{th} order statistics and k^{th} power of the s^{th} order statistics.

$$\nu_{r:n}^{(k)} = E\ (X_{r:n}^{k}) = \int_{-\infty}^{\infty} x^{k} f_{r:n}(x)\ dx \quad , \quad 1 \le r \le n$$

$$= C_{r:n} \int_{-\infty}^{\infty} x^{k}\ I\ (F\ ;\ r-1,\ n-r\ )\ d\ F(x) \qquad ...(1.3.1)$$

where

$$I(u;j,k) = u^j (1-u)^k , \quad 0 < u < 1 \qquad \ldots(1.3.2)$$

and

$$\nu^{(j,k)}_{r,s:n} = E(X^j_{r:n} \; X^k_{s:n})$$

$$= \iint_{x<y} x^j y^k \, f_{r,s:n}(x,y) \; dy \, dx$$

$$= C_{r,s:n} \iint_{x<y} x^j y^k \, I(F(x), F(y) \,;\, r-1, s-r-1, n-s) \, dF(x) \, dF(y) \qquad \ldots(1.3.3)$$

where

$$I(u,v;r,k,n) = u^r (v-u)^k (1-u)^n$$

$$\text{for} \quad r, s, k, n \ge 0 \quad \text{and} \quad 0 < u < v < 1 \qquad \ldots(1.3.4)$$

for nonnegative random variables we write

$$\nu^{(k)}_{r:n} = \int_0^\infty x^k \, f_{r:n}(x) \, dx \qquad \ldots(1.3.5)$$

$$= \int^\infty k \, x^{k-1} \left[1 - F_{r:n}(x) \right] dx \qquad \ldots(1.3.6)$$

1.4 OUTLIERS AND ITS RELATION WITH ORDER STATISTICS

In a sample taken from a certain population, it may appears that one or more than one values are surprisingly far away from the main group. These observations are known as outliers.

In other words, let us suppose $X_1, \ldots, X_n$ are i.i.d. random variables with cdf $F(x,\theta)$, where the parameter θ is possibly unknown. If this basic assumption is voilated in that one or possibly more than one of the X_i's are from a different population having cdf G which may or may not be completely specified from G are lebelled outliers or discordant observations.

RELATION OF OUTLIERS WITH ORDER STATISTICS

Outliers are to be found among the extremes of a data set. Extremes are examples of order statistics. It is thus relevant to ask to what extent the statistical methods of outliers and of order statistics coincide and depend on each other.

It is a general tendency in regarding the study of outliers as merely a subset of order statistics theory and method. After all, outliers are to be found among the class of sample extremes which are themselves particular forms of order statistics. But extreme values are not necessarily outliers and a substantial amount of outlier methodology make negligible direct appeal to behaviouristic properties of order statistics.

If the extremes $X_{(1)}$ or $X_{(n)}$ or both are unexpectedly extreme with respect to the cdf of random variables in the any sample

they may be called outliers or pair of outliers. Here we might wish to safeguard inferential studies against the prospect that $X_{(n)}$ is not representative of cdf.

The fundamental distinctions between the notation of an extreme and outlier and a contaminant readily show how tenuous is the link between outliers and order statistics. The extreme $X_{(n)}$ may or may not be an outlier depending on what may be reasonably expected under cdf. If $X_{(n)}$ declared an outlier, it may or may not be contaminant in the sense of alternative model.

The closest one comes to a direct link between order statistics and outliers in the modeling context is where the alternative model declares that $X_1 \ldots X_{n-1}$ are ordered random sample of size n-1 from F whilst there is single larger observation $X_{(n)}$ from a upwardly slipp e d distribution G. It has been employ in outliers study.

1.5 PROBABLITY DENSITY FUNCTION AND DISTRIBUTION FUNCTION FOR OUTLIERS

Let us represent the sample by n independent absolutely continuous random variables X_j (j = 1, 2, ..., n-1) and Y, such that X_j has pdf f(x) and cdf F(x) and Y has pdf g(x) and cdf G(x). Further, Let

$$Z_{1:n} \le Z_{2:n} \le \ldots \le Z_{n:n} \qquad \ldots(1.5.1)$$

be the order statistics obtained by arranging the n independent observations in increasing order of magnitude.

Then the pdf. of $Z_{r:n}$ ($1 \le r \le n$) is given by (David et al., 1977 ; David and Shu, 1978) as

$$\begin{aligned} h_{r:n}(x) = {} & \frac{(n-1)!}{(r-2)!(n-r)!} \Big[F(x) \Big]^{r-2} \Big[1 - F(x) \Big]^{n-r} G(x)\ f(x) \\ & + \frac{(n-1)!}{(r-1)!(n-r)!} \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r} g(x) \\ & + \frac{(n-1)!}{(r-1)!(n-r-1)!} \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r-1} \Big[1 - G(x) \Big] f(x) \\ & \qquad -\infty < x < \infty \qquad \ldots(1.5.2) \end{aligned}$$

where the first term drops out if r = 1 and the last term drops if

$r = n$.

The cdf. of $X_{r:n}$ is given by

$$H_{r:n}(x) = F_{r:n-1}(x) + \binom{n-1}{r-1} F^{r-1}(x) \left[1 - F(x) \right]^{n-r} G(x) ,$$

$$r = 1, \ldots, n-1$$

Therefore

$$H_{n:n}(x) = F^{n-1}(x)\, G(x) \qquad \ldots(1.5.3)$$

where $F_{r:n-1}(x)$ is the cdf of r^{th} order statistics in a sample of size $n-1$ in homogeneous case.

Similarly, the joint pdf. of $Z_{r:n}$ and $Z_{s:n}$ ($1 \le r < s \le n$) is given by (David et.al., 1977 ; David and Shu, 1978)

$$h_{r,s:n}(x) = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \left[F(x) \right]^{r-2} \left[F(y) - F(x) \right]^{s-r-1}$$

$$\left[1 - F(y) \right]^{n-s} G(x)\, f(x)\, f(y)$$

$$\frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!} \left[F(x) \right]^{r-1} \left[F(y) - F(x) \right]^{s-r-1}$$

$$\left[1 - F(y) \right]^{n-s} g(x)\, f(y)$$

$$\frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \left[F(x) \right]^{r-1} \left[F(y) - F(x) \right]^{s-r-2}$$

$$\left[1 - F(y) \right]^{n-s} \left[G(y) - G(x) \right] g(x)\, f(y)$$

$$\frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!} \left[F(x) \right]^{r-1} \left[F(y) - F(x) \right]^{s-r-1}$$

$$\left[1 - F(y) \right]^{n-s} f(x)\, g(y)$$

$$+ \quad \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!} \left[F(x) \right]^{r-1} \left[F(y) - F(x) \right]^{s-r-1}$$

$$\left[1 - F(y) \right]^{n-s-1} \left[1 - G(y) \right] f(x)\, f(y) \,, \qquad -\infty < x < y < \infty \qquad \ldots(1.5.4)$$

it can also be written as

$$h_{r,s:n}(x,y) \;=\; f_{r-1,s-1}(x,y)\; G(x) + f_{r,s:n-1}(x,y)\; (\; 1 - G(y) \;)$$

$$+ \frac{\;}{(r-1)!(s-r-1)!(n-s)!}\; F^{r-1}(x)\; (\; F(y) - F(x) \;)^{s-r-1} (\; 1 - F(y) \;)^{n-s}$$

$$\left[f(x)g(y) + g(x)f(y) + (s-r-1)\; \frac{G(y) - G(x)}{F(y) - F(x)}\; f(x)f(y) \right] \text{ for } \quad x < y$$

$$= \quad 0 \quad \text{elsewhere} \qquad \ldots(1.5.5)$$

where the first term drops out if $r = 1$, the last term drops if $s = n$, and the middle term drop if $s = r+1$.

1.6 SINGLE AND PRODUCT MOMENTS FOR OUTLIERS

The single and product moments of order statistics in the presence of outliers can be obtained as

$$\mu^{(k)}_{r:n} \;=\; E\;(\; Z^{k}_{r:n} \;) \;=\; \int_{-\infty}^{\infty} x^{k}\; h_{r:n}(x)\; dx \quad , \qquad 1 \le r \le n \qquad \ldots(1.6.1)$$

and

$$\mu_{r,s:n} \;=\; E\;(\; Z_{r:n} \quad Z_{s:n} \;) \;=\; \iint_{W_1} xy\; h_{r,s:n}(x,y)\; dx\; dy$$

$$1 \le r < s \le n \qquad \ldots(1.6.2)$$

where $w_1 = \left\{ (x,y) : -\infty < x < y < \infty \right\}$.

Let us denote the covariance between $Z_{r:n}$ and $Z_{s:n}$ by $\sigma_{r,s:n}$ ($1 \le r \le s$).

For the nonnegative random variables, we can write

$$\mu_{r:n}^{(k)} = \int_0^\infty x^k \, h_{r:n}(x) \, dx \qquad \ldots(1.6.3)$$

$$= \int_0^\infty k \, x^{k-1} \left[1 - H_{r:n}(x) \right] dx \qquad \ldots(1.6.4)$$

1.7 CONCEPT OF PERMANENT

Let S_n denote the set of permutations of 1, 2, ..., n . If A is an n x n matrix, then the permanent of A, denoted by per A, is defined as

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

where $a_{i\sigma(i)}$ is the element of matrix.

The permanent of a sqare matrix A is defined like the determinant, except that all signs are positive. It is usually written as $^{+}|A|^{+}$.

1.8 PROPERTIES OF PERMANENT

(1). The permanent clearly remains unchanged if the rows or columns of the matrix are permuted. Furthermore the permanent admits a Laplace expansion along any row or column of the matrix. Thus if we denote by A(i,j) the matrix obtained by deleting row i and column j of the n x n matrix A , then

$$\text{per } A = \sum_{j=1}^{n} a_{ij} \text{ per } A(i,j) \quad , \quad i = 1, 2, \ldots, n$$

and

$$\text{per } A = \sum_{i=1}^{n} a_{ij} \text{ per } A(i,j) \quad , \quad j = 1, 2, \ldots, n$$

(2). $A = (a_{ij})$ be an $n \times n$ real matrix, where first $n-1$ columns are nonnegative, then

$$(\text{per } (A))^2 \geq \text{per } (a_1, \ldots, a_{n-2}, a_{n-1}, a_{n-1}) \text{ per } (a_1, \ldots, a_{n-2}, a_n, a_n)$$

If $a_1, \ldots, a_{n-2}, a_{n-1}$ are positive, then equality holds in iff an is a multiple of a_{n-1}.

(3) Let $A = (a_{ij})$ be a column (or row) stochastic matrix satisfying

$$0 < \text{per } (A) \leq \text{per } (A (i /j))$$
$$i, j = 1, 2, \ldots, n \quad \text{then}$$

$$\text{per } (A (i/j)) = \text{per } A \quad ; \quad i, j = 1, 2, \ldots, n$$

(4). If A is a minimizing matrix in ψ_n , and B is the matrix obtained from A by replacing each of two arbitrary columns of A by their average, then

$$\text{per } (B) = \text{per } (A)$$

(5). If A is a minimizing matrix in ψ_n , then

$$\left[\text{per } (A) \right]^2 = \left[\sum_{i=1}^{n} a_{iq} \text{ per } (A (i/t)) \right] \left[\sum_{i=1}^{n} a_{it} \text{ per } (A (i/q)) \right]$$

for any q and t , $1 \leq q < t \leq n$

(6). If $A = (a_{ij})$ is an $n \times n$ complex matrix, then

$$\text{per}(A) = \sum_{k=0}^{n} (-1)^k \sum_{w \in q,n} \prod_{i=1}^{n} \left(\sum_{j=1} \lambda_j a_{ij} - a_{iw_1} - \cdots - a_{iw_k} \right)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are arbitrary complex numbers.

(7). If A and B are $n \times m$ complex matrices, $n \le m$, then

$$\text{per}(A) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{n!} \binom{n}{k} \text{per}(B + kA)$$

(8). If A and B are positive semi-definite hermitian $n \times n$ matrices, then

$$\text{per}(A*B) \le \text{per}(A)\,\text{per}(B)$$

and

$$\text{per}(A*A) \le (\text{per}(A))^2$$

(9). If A and $B = (b_{ij})$ are positive semidefinite hermitian $n \times n$ matrices, then

$$\text{per}(A*B) \le \text{per}(A) \prod_{i=1}^{n} (b_{ii})$$

(10). If $A = (a_{ij})$ and $B = (b_{ij})$ are positive semidefnite hermitian $n \times n$ matrices, then

$$\text{per}(A*B) + \text{per}(A)\,\text{per}(B) \ge \text{per}(A) \prod_{i=1}^{n} b_{ii} + \text{per} \prod_{i=1} a_{ii}$$

1.9 DISTRIBUTION FUNCTION AND PROBABLITY DENSITY FUNCTION IN THE FORM OF PERMANENT EXPRESSION

Let $X_1, X_2, \ldots, X_n$ be independent random variables having distribution functions $F_1(x)$, $F_2(x)$, ..., $F_n(x)$ and probablity density functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ respectively. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained by arranging the n X_i's in the increasing order of magnitude. Then the density function of $X_{r:n}$ $(1 \le r \le n)$, can be written in the form of permanents of matrices by Vaughan and Venables (1972) as follows

$$h_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \begin{array}{c} {}^{+}\left|\begin{array}{cccc} F_1(x) & F_2(x) & \cdots & F_n(x) \\ \cdot & \cdot & \cdots & \cdot \\ F_1(x) & F_2(x) & \cdots & F_n(x) \\ f_1(x) & f_2(x) & \cdots & f_n(x) \\ 1-F_1(x) & 1-F_2(x) & \cdots & 1-F_n(x) \\ \cdot & \cdot & \cdots & \cdot \\ 1-F_1(x) & 1-F_2(x) & \cdots & 1-F_n(x) \end{array}\right|^{+} \end{array} \begin{array}{l} r-1 \\ \text{rows} \\ \\ \\ \\ n-s \\ \text{rows} \end{array} \quad \ldots(1.9.1)$$

similarly joint density function is given as

$$h_{r,s:n}(x,y) = \frac{1}{(r-1)!(s-r-1)!(n-s)!}$$

$$ {}^{+}\left|\begin{array}{cccc} F_1(x) & F_2(x) & \cdots & F_n(x) \\ \cdot & \cdot & \cdots & \\ F_1(x) & F_2(x) & \cdots & F_n(x) \\ f_1(x) & f_2(x) & \cdots & f_n(x) \\ F_1(y)-F_1(x) & F_2(y)-F_2(x) & \cdots & F_n(y)-F_n(x) \\ \cdot & \cdot & \cdots & \cdot \\ F_1(y)-F_1(x) & F_2(y)-F_2(x) & \cdots & F_n(y)-F_n(x) \\ f_1(y) & f_2(y) & \cdots & f_n(y) \\ 1-F_1(y) & 1-F_2(y) & \cdots & 1-F_n(y) \\ \cdot & \cdot & \cdots & \cdot \\ 1-F_1(y) & 1-F_2(y) & \cdots & 1-F_n(y) \end{array}\right|^{+} \begin{array}{l} r-1 \\ \text{rows} \\ \\ \\ s-r-1 \\ \text{rows} \\ \\ \\ n-s \\ \text{rows} \\ \\ \end{array} \quad \ldots(1.9.2)$$

where $x < y$ and ${}^{+}|\, A \,|^{+}$ denote the permanent of matrix A.

Let us denote single moment of order statistics by $\mu_{r:n}^{(k)}$,$1 \le r \le n$ where

$$\mu_{r:n}^{(k)} = E (X_{r:n}^{k}) = \int_{-\infty}^{\infty} x^{k} \, h_{r:n}(x) \, dx \qquad \ldots(1.9.3)$$

similerly product moment of order statistics denoted by $\mu_{r,s:n}$, $1 \le r < s \le n$, where

$$\mu_{r,s:n} = E (X_{r:n} \; X_{s:n}) = \iint_{-\infty < x < y < \infty} x \; h_{r,s:n}(x,y) \, dy \, dx \qquad \ldots(1.9.4)$$

1.10 TRUNCATION

Statistical problem of truncation arise when a standard statistical model is appropriate for analysis except that values of the random variable falling below or above some value are not measured at all. For example, in a study of particle size, particles below the resolving power of observational equipment will not be seen at all. Most of the existing theory for problems of this sort takes the limits at which truncation occurs to be known constant. But there are practical situations in which these limits are not exactly known. Truncation is sometimes usefully regarded as a special case of selection.

Particularly , if values below a certain limit, a , are not observed at all, the distribution is said to be truncated on left . If the values larger than an upper limit , b , are not observed, the distribution is said to be truncated on the right. If only values lying between a and b are observed the distribution is said to be double truncated.

The pdf of truncated model has defined in Khan *et al.* (1983). If we represent the truncation points by Q_1 and P_1 at left and right respectively, then in the case of doubly truncated model, the pdf is given as

$$\frac{f(x)}{P - Q}, \qquad Q_1 \le x \le P_1 \qquad \ldots(1.10.1)$$

where

$$\int_{-\infty}^{Q_1} f_1(x)\, dx = Q \qquad \ldots(1.10.2)$$

and

$$\int_{-\infty}^{P_1} f(x)\, dx = P \qquad \ldots(1.10.3)$$

and Q and (1 − P) are respectively the portion of truncation on the left and right of the distribution. P and Q assumed to be known ($Q < P$) and Q_1 and P_1 are functions of Q and P.

Distribution of truncated order statistics

(a) Left trucated at x

Let $Q = F(x)$, $P = 1$, then the truncated distribution has pdf

$$\frac{f(t)}{1 - F(x)}, \qquad x < t < \infty$$

and the pdf of $X_{r:n} = y$ in this case, will be

$$C_{r:n}\left[\frac{F(y) - F(x)}{1 - F(x)}\right]^{r-1}\left[\frac{1 - F(y)}{1 - F(x)}\right]^{n-r}\frac{f(y)}{1 - F(x)} \qquad \ldots(1.10.4)$$

(b) Right truncated at y

Similarly at $Q = 0$, $P = F(y)$, then the truncated distribution has pdf

$$\frac{f(t)}{F(y)} , \quad -\infty < t < y$$

and the pdf of $X_{r:n} = x$ will be

$$C_{r:n} \left[\frac{F(x)}{F(y)}\right]^{r-1} \left[\frac{F(y) - F(x)}{F(y)}\right]^{n-r} \frac{f(x)}{F(y)} \qquad \ldots(1.10.5)$$

Chapter-II

RECUERRENCE RELATIONS OF MOMENTS OF ORDER STATISTICS IN THE PRESENCE OF ONE OUTLIER

2.1 INTRODUCTION

Here, we review several relations and identities obtained by Balakrishnan (1987, 1988), which satisfy the single and product moments of order statistics from a sample of size n in the presence of an outlier. These identities generalize the result of Joshi(1973). Balakrishnan(1988) has shown that it is sufficient to evaluate at most two single moments and (n-2)/2 product moments when n is even and two single moments and (n-1)/2 product moments when n is odd. These generalize the results of Govindarajulu(1963), Joshi(1971), Joshi and Balakrishnan(1982) to the case when sample includes an outlier. Balakrishnan(1988) also established some single identities involving linear combination of co-variance of order statistics which minimize the numerical calculations considerably. Here we also review some recurrence relations among the single and product moments of order statistics from sample size n from right truncated exponential distribution in the presence of an outlier obtained by Shubha and Joshi(1991).

2.2 IDENTITIES FOR SINGLE MOMENTS

Joshi (1973) has established the following two identities for moments of order statistics,

$$\sum_{r=1}^{n} \frac{1}{r}\,\nu_{r:n}^{(k)} = \sum_{r=1}^{n} \frac{1}{r}\,\nu_{1:r}^{(k)} \qquad \ldots(2.2.1)$$

$$\sum_{r=1}^{n} \frac{1}{n-r+1}\,\nu_{r:n} = \sum_{r=1}^{n} \frac{1}{n}\,\nu_{r:r}^{(k)} \qquad \ldots(2.2.2)$$

Balakrishnan (1987) generalized the results of Joshi(1973) for the case of order statistics in the presence of an outlier as given in Identity 2.2.1 & 2.2.2.

IDENTITY 2.2.1:

For $n \geq 2$

$$\sum_{r=1}^{n} \frac{1}{r}\, h_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r}\, f_{r:n}(x) + \frac{1}{n} \sum_{r=1}^{n} \left\{ h_{1:r}(x) - f_{1:r}(x) \right\}$$

$$= \frac{1}{n} \sum_{r=1}^{n} h_{1:r}(x) + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] f_{1:r} \qquad \ldots(2.2.3)$$

PROOF: From (1.5.2), we have

$$\sum_{r=1}^{n} \frac{1}{r}\, h_{r:n}(x) = I_1 + I_2 + I_3 \qquad \ldots(2.2.4)$$

where

$$I_1 = \sum_{r=2}^{n} \frac{(n-1)!}{r!(n-r)!}\,(r-1) \Big[F(x) \Big]^{r-2} \Big[1 - F(x) \Big]^{n-r} G(x)f(x)$$

$$= \frac{1}{F(x)} \sum_{r=1}^{n-1} \binom{n-1}{r} \Big[F(x) \Big]^{r} \Big[1 - F(x) \Big]^{n-1-r} G(x)f(x)$$

$$- \frac{1}{n\Big[F(x)\Big]^2} \sum_{r=1}^{n} \binom{n}{r} \Big[F(x) \Big]^{r} \Big[1 - F(x) \Big]^{n-r} G(x)f(x)$$

$$= \sum_{r=0}^{n-2} \Big[1 - F(x)\Big]^{r} G(x)f(x) - \frac{1}{n} \sum_{r=0}^{n-2} (r+1) \Big[1 - F(x)\Big]^{r} G(x)f(x)$$

$$I_2 = \sum_{r=1}^{n} \frac{(n-1)!}{r!(n-r)!} \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r} g(x)$$

$$= \frac{1}{n\,F(x)} \sum_{r=1}^{n} \binom{n}{r} \left[F(x)\right]^{r} \left[1 - F(x)\right]^{n-r} g(x)$$

$$= \frac{1}{n} \sum_{r=0}^{n-1} \left[1 - F(x)\right]^{n-r} g(x)$$

and

$$I_3 = \sum_{r=1}^{n-1} \frac{(n-1)!}{r!\,(n-r-1)!} \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r-1} \left[1-G(x)\right] f(x)$$

$$= \frac{1}{F(x)} \sum_{r=1}^{n-1} \binom{n-1}{r} \left[F(x)\right]^{r} \left[1 - F(x)\right]^{n-r-1} \left[1-G(x)\right] f(x)$$

$$= \sum_{r=0}^{n-2} \left[1 - F(x)\right]^{r} \left[1-G(x)\right] f(x)$$

Now substituting these expression for I_1, I_2, I_3 in (2.2.4) and obtain the identity (2.2.3).

IDENTITY 2.2.2:

For $n \geq 2$

$$\sum_{r=1}^{n} \frac{1}{n-r+1} h_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{n-r+1} f_{r:n}(x) + \frac{1}{n} \sum_{r=1}^{n} \left\{ h_{r:r}(x) - f_{r:r}(x) \right\}$$

$$= \frac{1}{n} \sum_{r=1}^{n} h_{r:r}(x) + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] f_{r:r}(x) \qquad \ldots(2.2.5)$$

In terms of moments, these results yield the following Identities,

$$\sum_{r=1}^{n} \frac{1}{r} \mu_{r:n}^{(k)} = \frac{1}{n} \sum_{r=1}^{n} \mu_{1:r}^{(k)} + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] \nu_{1:r}^{(k)} \qquad \ldots(2.2.6)$$

and

$$\sum_{r=1}^{n} \frac{1}{n-r+1} \mu_{r:n}^{(k)} = \frac{1}{n} \sum_{r=1}^{n} \mu_{r:r}^{(k)} + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] \nu_{r:r}^{(k)}$$

2.3 RECURRENCE RELATIONS FOR SINGLE MOMENTS IN THE PRESENCE OF AN OUTLIER

THEOREM 2.3.1: (Balakrishnan, 1988)

For $1 \le r \le n-1$ and $k \ge 1$

$$r\,\mu^{(k)}_{r+1:n} + (n-r)\mu^{(k)}_{r:n} = (n-1)\mu^{(k)}_{r:n-1} + \nu^{(k)}_{r:n-1} \qquad \ldots(2.3.1)$$

PROOF: Let us suppose,

$$I_1 = r\,\mu^{(k)}_{r+1:n}$$

$$I_2 = (n-r)\mu^{(k)}_{r:n}$$

$$I_3 = (n-1)\mu^{(k)}_{r:n-1}$$

$$I_4 = \nu^{(k)}_{r:n-1}$$

Now simplify I_1 as

$$I_1 = r\,\mu^{(k)}_{r+1:n}$$

$$= (r-1)\,\mu^{(k)}_{r+1:n} + \mu^{(k)}_{r+1:n}$$

$$I_1 = I_{11} + I_{12}$$

where

$$I_{11} = (r-1)\int_{-\infty}^{\infty} x^k\, h_{r+1:n}(x)\,dx$$

$$= \frac{(r-1)(n-1)!}{(r-1)!(n-r-1)!}\int_{-\infty}^{\infty} x^k \Big[F(x)\Big]^{r-1}\Big[1-F(x)\Big]^{n-r-1} G(x)f(x)\,dx$$

$$+ \frac{(r-1)(n-1)!}{(r)!(n-r-1)!}\int_{-\infty}^{\infty} x^k \Big[F(x)\Big]^{r}\Big[1-F(x)\Big]^{n-r-1} g(x)\,dx$$

$$+ \frac{(r-1)(n-1)!}{(r)!(n-r-2)!}\int_{-\infty}^{\infty} x^k \Big[F(x)\Big]^{r}\Big[1-F(x)\Big]^{n-r-2}(1-G(x))\,f(x)\,dx \qquad \ldots(2.3.2)$$

and

$$I_{12} = \int_{-\infty}^{\infty} x^k \; h_{r+1:n}(x) \; dx$$

$$= \frac{(\ \ n-1)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r-1} G(x)f(x) \; dx$$

$$+ \frac{(n-1)!}{(r)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r} \left[1 - F(x) \right]^{n-r-1} g(x) \; dx$$

$$+ \frac{(n-1)!}{r!(n-r-2)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r} \left[1 - F(x) \right]^{n-r-2} (1-G(x)) \; f(x) \; dx$$

...(2.3.3)

$$I_2 = (n-r)\mu_{r:n}^{(k)}$$

$$= (n-r-1) \; \mu_{r:n}^{(k)} \longrightarrow \mu_{r:n}^{(k)}$$

$$I_2 = \qquad I_{21} \qquad\qquad I_{22}$$

Now

$$I_{21} = (n-r-1) \int_{-\infty}^{\infty} x^k \; h_{r:n}(x) \; dx$$

$$= \frac{(n-r-1)(n-1)!}{(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-2} \left[1 - F(x) \right]^{n-r} G(x)f(x) \; dx$$

$$+ \frac{(r-1)(n-1)!}{(r)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r} g(x) \; dx$$

$$+ \frac{(r-1)(n-1)!}{(r)!(n-r-2)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r} \left[1 - F(x) \right]^{n-r-2} (1-G(x)) \; f(x) \; dx$$

...(2.3.4)

$$I_{22} = \int_{-\infty}^{\infty} x^k \; h_{r:n}(x) \; dx$$

$$= \frac{(n-1)!}{(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-2} \left[1 - F(x) \right]^{n-r} G(x)f(x) \; dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r} g(x) \; dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r-1} (1-G(x)) \; f(x) \; dx$$

...(2.3.5)

Now adding and simplifying above equations

$$I_1 + I_2 = \frac{(n-1)(n-2)!}{(r-2)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \Big[F(x) \Big]^{r-2} \Big[1 - F(x) \Big]^{n-r-1} G(x) f(x) dx$$

$$+ \frac{(n-1)(n-2)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r-1} g(x)\, dx$$

$$+ \frac{(n-1)(n-2)!}{(r-1)!(n-r-2)!} \int_{-\infty}^{\infty} x^k \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r-2} (1-G(x)) f(x) dx$$

$$+ \frac{(n-2)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \Big[F(x) \Big]^{r-1} \Big[1 - F(x) \Big]^{n-r-1} f(x)\, dx$$

$$= I_3 + I_4$$

$$= (n-1)\, \mu_{r:n-1}^{(k)} + \nu_{r:n-1}^{(k)}$$

The theorem is proved.

THEOREM 2.3.2: (Balakrishnan, 1988)

For $1 \le r \le n-1$ and $k \ge 1$

$$\mu_{r:n}^{(k)} = \sum_{j=r}^{n} (-1)^{j-r} \binom{n-1}{j-1} \binom{j-1}{r-1} \mu_{j:j}^{(k)} + \nu_{r:n-1}^{(k)} \qquad \ldots(2.3.6)$$

PROOF: First with the help of (1.5.2) and (1.6.1) we write the k^{th} moment,

$$\mu_{r:n}^{(k)} = \int_{-\infty}^{\infty} x^k \, h_{r:n}(x)\, dx$$

$$= \frac{(n-1)!}{(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-2} \left[1 - F(x)\right]^{n-r} G(x)f(x)\, dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} g(x)\, dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r-1} (1-G(x))\, f(x)\, dx$$

In the first part of expression, we expand $\left[1 - F(x)\right]^{n-r}$ binomially in the powers of $F(x)$. And in the third part of expression, we expand $\left[1 - F(x)\right]^{n-r-1}$ binomially in the powers of $F(x)$. Thereby after calculation we would get the RHS of (2.3.6)

THEOREM 2.3.3: (Balakrishnan, 1988)

For $2 \le r \le n$ and $k \ge 1$,

$$\mu_{r:n}^{(k)} = \sum_{j=r-1}^{n} (-1)^{j-n+r-1} \binom{n-1}{j-1} \binom{j-1}{n-r} \mu_{1:j}^{(k)} + \nu_{r-1:n-1}^{(k)} \qquad \ldots(2.3.7)$$

PROOF: We write the expression of k^{th} moment with the help of (1.5.2) and (1.6.1)

$$\mu_{r:n}^{(k)} = \int_{-\infty}^{\infty} x^k \, h_{r:n}(x)\, dx$$

$$= \frac{(n-1)!}{(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-2} \left[1 - F(x)\right]^{n-r} G(x)f(x)\, dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} g(x)\, dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r-1} (1-G(x))f(x)\, dx$$

$$= \frac{(n-1)!}{(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[1-(1-F(x))\right]^{r-2} \left[1-F(x)\right]^{n-r} G(x)f(x)\,dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k \left[F(x)\right]^{r-1} \left[1-F(x)\right]^{n-r} g(x)\,dx$$

$$+ \frac{(n-1)!}{(r-1)!(n-r-1)!} \int_{-\infty}^{\infty} x^k \left[1-(1-F(x))\right]^{r-1} \left[1-F(x)\right]^{n-r-1} (1-G(x))f(x)\,dx \qquad \ldots(2.3.8)$$

In (2.3.8), we expand term $\left[1-(1-F(x))\right]^{r-2}$ binomially in the $(1-F(x))$ and term $\left[1-(1-F(x))\right]^{r-1}$ binomially in the $(1-F(x))$ then after calculation we will find RHS of (2.3.7).

There are some specific recurrence relations for single moments of the truncated exponential model established by Shubha and Joshi(1991). Here he define cdf and pdf as below

$$\left.\begin{aligned} F_0 &= 1 - \exp(-x_0) \\ G_0 &= 1 - \exp(-\alpha\, x_0) \\ f(x) &= e^{-x}/\,F_0\,, \quad 0 < x \le x_0 \\ g(x) &= \alpha\, e^{-\alpha x}/\,G_0\,, \quad 0 < x \le x_0 \end{aligned}\right\} \qquad \ldots(2.3.9)$$

where $\alpha > 0$ and the truncation point x_0 is fixed and assumed to be known. These results are generalization of results of Joshi(1978).

THEOREM 2.3.4: (Shubha and Joshi, 1991)

For $k = 1, 2, \ldots$ and $r = 1, 2, \ldots, n-1$

$$\mu_{r:n}^{(k)} = \frac{1}{(n-1+\alpha)} \left[\frac{(n-1)}{F_0} \left(\mu_{r-1:n-1}^{(k)} - e^{-x_0} \mu_{r:n-1}^{(k)} \right) + \frac{\alpha}{G_0} \left(\nu_{r-1:n-1}^{(k)} - e^{-\alpha x_0} \nu_{r:n-1}^{(k)} \right) + k\, \mu_{r:n}^{(k+1)} \right] \qquad \ldots(2.3.10)$$

where $\mu^{(0)}_{r:n} = 1, \quad 1 \le r < n$

$$\mu^{(k)}_{0:t} = 0, \quad k = 1, 2, \ldots \quad t = 0, 1, 2, \ldots$$

$$\nu^{(k)}_{0:t} = 0, \quad k = 1, 2, \ldots \quad t = 0, 1, 2, \ldots$$

PROOF: Using (1.5.3), (1.6.4) and (1.3.6), we obtain

$$\mu^{(k)}_{r:n} = \nu^{(k)}_{r:n-1} - I_1 + I_2 \qquad \ldots(2.3.11)$$

where $I_1 = \binom{n-1}{r-1} \int_0^{x_0} k\, x^{k-1} \left(\frac{1-e^{-x}}{F_0} \right)^{r-1} \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-r} \frac{1}{G_0}\, dx$

$$I_2 = \binom{n-1}{r-1} \frac{1}{G_0} \int_0^{x_0} k\, x^{k-1} \left(\frac{1-e^{-x}}{F_0} \right)^{r-1} \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-r} e^{-\alpha x}\, dx$$

$$\ldots(2.3.12)$$

Now integrate I_1 by treating x^{k-1} for integration and $\left(\frac{1-e^{-x}}{F_0} \right)^{r-1} \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-r}$ for differentiation and using (1.3.5) we get the the value of I_1. Similarly Integerate I_2 by treating $e^{-\alpha x}$ for integeration and $\left(\frac{1-e^{-x}}{F_0} \right)^{r-1} \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-r}$ for differential and get the value of I_2 and using result obtained by Joshi(1978)

$$\nu^{(k)}_{r:n-1} = \frac{1}{F_0} \nu^{(k)}_{r-1:n-2} + \frac{k}{n-1} \nu^{(k-1)}_{r:n-1} - \frac{e^{-x_0}}{F_0} \nu^{(k)}_{r:n-2}, \quad 1 \le r < n-2$$

$$\ldots(2.3.13)$$

Adding (2.3.11), (2.3.12) and (2.3.13) we get the required result.

THEOREM 2.3.5: (Shubha and Joshi, 1991)

For $k = 1, 2, \ldots,$

$$\mu_{n:n}^{(k)} = \frac{1}{n-1+\alpha}\left[\frac{(n-1)}{F_0}\left(\mu_{n-1:n-1}^{(k)} - x_0^k \; e^{-x_0}\right) + k\,\mu_{n:n}^{(k)}\right.$$

$$\left. - \frac{\alpha}{G_0}\left(\nu_{n-1:n-1}^{(k)} - x_0^k\, e^{-\alpha x_0}\right)\right] \qquad \ldots(2.3.14)$$

PROOF: Its proof is analogous to previous Theorem.

COROLLARY 2.3.1: For samples containing a an outlier from an exponential distribution

$$\mu_{r:n}^{(k)} = \frac{1}{n-1+\alpha}\left[(n-1)\,\mu_{r-1:n-1}^{(k)} + \alpha\,\nu_{r-1:n-1}^{(k)} + k\,\mu_{r:n}^{(k-1)}\right] \qquad \ldots(2.3.15)$$

PROOF: Proof follows immediately on taking limit as $x_0 \to \infty$ in (2.3.10) & (2.3.14).

2.4 RECURRENCE RELATIONS FOR PRODUCT MOMENTS IN THE PRESENCE OF AN OUTLIER

THEOREM 2.4.1 (Balakrishnan, 1988)

For $2 \le r < s \le n$

$$(r-1)\mu_{r,s:n} + (s-r)\mu_{r-1,s:n} + (n-s+1)\mu_{r-1,s-1:n}$$

$$= (n-1)\mu_{r-1,s-1:n-1} + \nu_{r-1,s-1:n-1} \qquad \ldots(2.4.1)$$

PROOF: With the help of (1.5.4) and (1.6.2), we obtain the expression of LHS of (2.4.1). Now split first term in $(r-1)\mu_{r,s:n}$ into two by writing the multiple $(r-1)$ as $((r-2)+1)$ then split the middle term in $(s-r)\mu_{r-1,s:n}$ into two by writing the multiple $(s-r)$ as $((s-r-1)+1)$ and similarly split the last term in $(n-s+1)$ as $((n-s)+1)$. Now adding all three expressions and simplifying, we obtain the RHS of (2.4.1).

Denote $W_1 = \left\{ (x,y) : -\infty < x < y < \infty \right\}$

$W_2 = \left\{ (x,y) : -\infty < y < x < \infty \right\}$, we write from (1.5.4) and (1.2.7) as

$$\mu_{r,s:n} = \iint_{W_2} xy\, h_{r,s:n}(y,x)\, dx\, dy \qquad 1 \le r < s \le n \qquad \text{...(2.4.2)}$$

and

$$\nu_{r,s:n} = \iint_{W_2} xy\, f_{r,s:n}(y,x)\, dx\, dy \qquad 1 \le r < s \le n \qquad \text{...(2.4.3)}$$

Noting that

$$W_1 \cup W_2 = R^2 = \left\{ (x,y),\ -\infty < x < \infty\ ,\ -\infty < y < \infty \right\},$$

THEOREM 2.4.2: (Balakrishnan, 1988)

For arbitrary continuous cdf's $F(x)$ and $G(x)$ and for $1 \le r < s \le n$,

$$\mu_{r,s:n} - \nu_{r,s:n} + \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{n-1}{j} \binom{n-j-1}{k} \left\{ \mu_{n-s-j+1,n-r-j+1:n-j-k} - \nu_{n-s-j+1,n-r-j+1:n-j-k} \right\}$$

$$= \frac{1}{n} \sum_{j=1}^{s-r} (-1)^{s-r-j} \binom{n}{s-j} \binom{s-j-1}{r-1} \Big\{ (s-j)\, \nu_{j:n-s+j} (\mu_{s-j:s-j} - \nu_{s-j:s-j}) + (n-s+j)\, \nu_{s-j:s-j} (\mu_{j:n-s+j} - \nu_{j:n-s+j}) \Big\}$$

...(2.4.4)

PROOF: For $1 \le r < s \le n$, consider

$$I = \iint_{R^2} xy\, h_{r,s:n}(y,x)\, dx\, dy$$

$$= \iint_{w_1} xy\, h_{r,s:n}(y,x)\, dx\, dy + \iint_{w_2} xy\, h_{r,s:n}(y,x)\, dx\, dy$$

$$= \mu_{r,s:n} + \iint_{w_2} xy\, h_{r,s:n}(y,x)\, dx\, dy$$

Now expanding the terms $\left\{ F(x) \right\}^a = \left\{1-(1-F(x)\right\}^a$ and $\left\{1 - F(y)\right\}^b$ binomially in powersof $(1 - F(x))$ and $F(y)$, respectively in the integral over w_2 and simplifying the resulting expression using (2.4.2) & (2.4.3), we get

$$I = \mu_{r,s:n} + \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{n}{j} \binom{n-j}{k} \nu_{n-s-j+1,n-r-j+1:n-j-k}$$

$$+ \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{n-1}{j} \binom{n-1-j}{k}$$

$$\left\{ \mu_{n-s-j+1,n-r-j+1:n-j-k} - \nu_{n-s-j+1,n-r-j+1:n-j-k} \right\}$$

which, upon using the result of Joshi and Balakrishnan(1982) yields

$$I = \mu_{r,s:n} - \nu_{r,s:n} + \sum_{j=1}^{s-r} (-1)^{s-r-j} \binom{n}{s-j} \binom{s-1-j}{r-1} \nu_{s-j:s-j}\, \nu_{j:n-s+j}$$

$$+ \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-k-j} \binom{n-1}{j} \binom{n-1-j}{k}$$

$$\left\{ \mu_{n-s-j+1,n-r-j+1:n-j-k} - \nu_{n-s-j+1,n-r-j+1:n-j-k} \right\} \qquad \ldots(2.4.5)$$

We can also write

$$I = \iint_{R^2} xy\, h_{r,s:n}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, h_{r,s:n}(x,y)\, dx\, dy$$

Now expanding the term $\left\{ F(y) - F(x) \right\}^{a}$ binomially in powers of $F(x)$ and $F(y)$ and simplifying the resulting expression using (1.6.2) and (1.3.4), they obtain

$$I = \frac{1}{n} \sum_{j=1}^{s-r} (-1)^{s-r-j} \binom{n}{s-j} \binom{s-1-j}{r-1} \left\{ (s-j)\, \nu_{j:n-s+j}\, \mu_{s-j:s-j} + (n-s+j)\, \nu_{s-j:s-j}\, \mu_{j:n-s+j} \right\} \quad \ldots(2.4.6)$$

Relation of (2.4.4) follows immediately upon equating (2.4.5) and (2.4.6).

It should be noted that relation (2.4.4) contains only two product moments viz $\mu_{r,s:n}$ and $\mu_{n-s+1,n-r+1:n}$ in a samples of size n from the outlier model. In particular, for s=r+1, we have the following corollaries.

COROLLARY 2.4.1 For $r = 1, 2, \ldots\ldots, n-1$,

$$\mu_{r,r+1:n} - \nu_{r,r+1:n} + (-1)^{n} \left\{ \mu_{n-r,n-r+1:n} - \nu_{n-r,n-r+1:n} \right\}$$
$$= \sum_{j=1}^{n-r-1} \sum_{k=0}^{r-1} (-1)^{n+1-j-k} \binom{n-1}{j} \binom{n-1-j}{k}$$
$$\left\{ \mu_{n-r-j,n-r-j+1:n-j-k} - \nu_{n-r-j,n-r-j+1:n-j-k} \right\} + \sum_{k=1}^{r-1} (-1)^{n+1-k} \binom{n-1}{k}$$
$$\left\{ \mu_{n-r,n-r+1:n-k} - \nu_{n-r,n-r+1:n-k} \right\} + \frac{1}{n} \binom{n}{r} \left\{ r\, \nu_{1:n-r} (\mu_{r:r} - \nu_{r:r}) + (n-r)\, \nu_{r:r} (\mu_{1:n-r} - \nu_{1:n-r}) \right\} \quad \ldots(2.4.7)$$

Similarly for s=n−r+1 , we have the following corollary.

COROLLARY 2.4.2 For r = 1, 2, ..., (n/2)

$$\left\{ 1 + (-1)^n \right\} \left\{ \mu_{r,n-r+1:n} - \nu_{r,n-r+1:n} \right\}$$

$$= \sum_{j=1}^{r-1} \sum_{k=0}^{r-1} (-1)^{n+1-j-k} \binom{n-1}{j} \binom{n-1-j}{k}$$

$$\left\{ \mu_{r-j,n-r-j+1:n-j-k} - \nu_{r-j,n-r-j+1:n-j-k} \right\}$$

$$+ \sum_{k=1}^{r-1} (-1)^{n+1-k} \binom{n-1}{k} \left\{ \mu_{r,n-r+1:n-k} - \nu_{r,n-r+1:n-k} \right\}$$

$$+ \frac{1}{n} \sum_{j=1}^{n-2r+1} (-1)^{n+1-j} \binom{n}{r+j-1} \binom{n-r-j}{r-1}$$

$$\Big\{ (r+j-1)\, \nu_{n-r-j+1:n-r-j+1} \left(\mu_{j:r+j-1} - \nu_{j:r+j-1} \right) +$$

$$(n-r-j+1)\nu_{j:r+j-1}(\mu_{n-r-j+1:n-r-j+1} - \nu_{n-r-j+1:n-r-j+1}) \Big\} \quad \ldots(2.4.8)$$

Corollary(2.4.2) shows that if n is even then product moments $\mu_{r,n-r+1}$ for $1 \le r \le (n/2)$ can all be obtained from the moments in samples of sizes (n−1) and less. For example, for r = 1 and even values of n, they obtain the relation

$$2 \left\{ \mu_{1,n:n} - \nu_{1,n:n} \right\} = \frac{1}{n} \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n}{j} \Big\{ j\, \nu_{n-j:n-j} \left(\mu_{j:j} - \nu_{j:j} \right)$$

$$+ (n-j)\, \nu_{j:j} \left(\mu_{n-j:n-j} - \nu_{n-j:n-j} \right) \Big\} \quad \ldots(2.4.9)$$

which, upon using the result that (Govindarajalu,1963;Joshi and Balakrishnan, 1982)

$$2\,\nu_{1,n:n} = \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n}{j} \nu_{j:j}\, \nu_{n-j:n-j}$$

and simplifying yields the relation

$$\mu_{1,n:n} = \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n-1}{j} \nu_{j:j}\, \mu_{n-j:n-j}$$

for the even value of n.

Govindarajalu(1963) and Joshi and Balakrishnan(1982) have obtained upper bounds for the number of single and product moments to be evaluated for calculating all moments $\nu_{r:n}$, $\nu^{(2)}_{r:n}$ and $\nu_{r,s:n}$ provided moments are available in samples of sizes (n−1) and less. Making use of corollaries (2.4.1) and (2.4.2), They obtain in the following theorem similar upper bonds for the number of single and product moments to be evaluated for calculating all the moments $\mu_{r:n}$, $\mu^{(2)}_{r:n}$ and $\mu_{r,s:n}$ provided these moments are available in the samples of sizes n−1 and less.

THEOREM 2.4.3: (Balakrishnan, 1988)

In order to find the first two single moments and product moments of order statistics in a sample of size n involving a single outlier, given these moments and also the moments from the population with cdf F(x) in samples of size n−1 and less, one has to evaluate at most single moments (n−2)/2 product moment if n is even and two single moments and (n−1)/2 product moments if n is odd.

PROOF: With the help of any relation of Theorems(2.3.1) to (2.3.3) and with the help of Theorem(2.4.1), we can easily evaluate just two single moments ($\mu_{n:n}$, $\mu^{(2)}_{n:n}$) for calculating $\mu_{r:n}$ and $\mu^{(2)}_{r:n}$

and just $(n-1)$ product moments ($\mu_{r,r+1:n}$, $1 \le r \le n-1$) for calculating all product moments $\mu_{r,s:n}$ ($1 \le r < s \le n$). However, when n is odd, we need to calculate only $(n-1)/2$ product moments $\mu_{r,r+1:n}$ ($1 \le r \le (n-1)/2$) as the remaining $(n-1)/2$ product moments $\mu_{r,r+1:n}$ ($(n+1)/2 \le r \le n-1$) can be obtained from corollary(2.3.1). Similarly , when n is even, say $n = 2m$, we need to calculate only $(n-2)/2 = m-1$ product moments $\mu_{1,2:2m}$, $\mu_{2,3:2m}$, ..., $\mu_{m-1,m:2m}$, since $\mu_{m,m+1:2m}$ can be obtained from corollary (2.3.2). Where as the product moments $\mu_{r,r+1:2m}$ ($m+1 \le r \le 2m-1$) can all be obtained from corollary(2.4.1) which prove the theorem.

Now we review some specific recurrence relations for product moments of the truncated exponential model given by (Shubha and Joshi, 1991).

THEOREM 2.4.4: (Shubha and Joshi, 1991)

For $n = 3, 4, \ldots..$

$$\mu_{1,2:n} = \frac{1}{n-2+\alpha}\Bigg[\mu_{1:n} + \frac{(n-1)(1-\alpha)}{2\alpha}\mu^{(2)}_{1:n} + \frac{e^{-x_0}}{F_0}\frac{(n-1)(\alpha+1)}{2\alpha}\mu^{(2)}_{1:n-1} - \frac{e^{-x_0}}{F_0}(n-1)\,\mu_{1,2:n-1} - \frac{\alpha\, e^{-\alpha x_0}}{G_0}\nu_{1,2:n-1} - \frac{(1-\alpha)}{\alpha\, G_0}\nu_{1:n-1} + \frac{e^{-\alpha x_0}}{G_0}\left(1 + \frac{(n-1)(1-\alpha)}{2\alpha}\right)\nu^{(2)}_{1:n-1} + \frac{(n-1)(1-\alpha)}{2\alpha}\frac{e^{-x_0}}{F_0}\frac{e^{-\alpha x_0}}{G_0}\nu^{(2)}_{1:n-2}\Bigg] \quad \ldots(2.4.10)$$

For $n \ge 2$

$$\mu_{n-1,n:n} = \mu_{n-1:n} - (n-1)\,x_0\,\frac{e^{-x_0}}{F_0}\mu_{n-1:n-1} + \frac{(n-1)}{F_0}\mu^{(2)}_{n-1:n-1}\frac{(\alpha+1)}{\alpha} - \frac{(n-1)(\alpha+1)}{2\alpha}\mu^{(2)}_{n:n} - \frac{\nu_{n-1:n-1}}{G_0}\left[e^{-\alpha x_0}\left(\frac{1}{\alpha} + x_0\right) - \frac{(1-\alpha)}{\alpha}\right]$$

$$+ \frac{\nu_{n-1:n-1}}{G_0} - \frac{(1-\alpha)(n-1)e^{-x_0}\, x_0^2}{2\alpha\, F_0} \qquad \ldots(2.4.11)$$

where notations are given in (2.3.9)

PROOF: The method used in proving these results is analogous to the one given by Joshi(1982). they write

$$\mu_{1:n} = E(\, X_{1:n} \quad X^0_{2:n})$$

from (1.5.5) and (1.6.2), It means

$$\mu_{1:n} = (n-1)(n-2)\iint_W x\ (1 - \frac{1-e^{-y}}{F_0})^{n-3}\ (1 - \frac{1-e^{-\alpha y}}{G_0})\frac{e^{-x}}{F_0}\ \frac{e^{-y}}{F_0}\ dx\ dy$$

$$+ (n-1)\Big[\iint_W x\ (1 - \frac{1-e^{-y}}{F_0}\ )^{n-2}\ \frac{e^{-x}}{F_0}\ \frac{\alpha e^{-\alpha y}}{G_0}\ dx\ dy$$

$$+ \iint_W x\ (1 - \frac{1-e^{-y}}{F_0}\ )^{n-2}\ \frac{e^{-y}}{F_0}\ \frac{\alpha e^{-\alpha x}}{G_0}\ dx\ dy\ \Big]$$

For simplifying $\mu_{1:n}$ as

$$\mu_{1:n} = I_1 + I_2 + I_3 \qquad \ldots(2.4.12)$$

Now consider these integrals separately.

$$I_1 = (n-1)(n-2)\int_0^{x_0} x\ \frac{e^{-x}}{F_0}\ I_4\ dx$$

where $$I_4 = \int_0^{x_0} (\ 1 - \frac{1-e^{-y}}{F_0}\ )^{n-3}(\ 1 - \frac{1-e^{-\alpha y}}{G_0}\ )\frac{e^{-y}}{F_0}\ dy,$$

Now integrate by parts by treating 1 for integration and $(\ 1 - \frac{1-e^{-y}}{F_0}\ )^{n-3}(\ 1 - \frac{1-e^{-\alpha y}}{G_0}\ )\frac{e^{-y}}{F_0}$ for differentiation and substitute the value of I_4 in I_1 , Therefore

$$I_1 = (n-1)(n-2)\Big[\ - \int_0^{x_0} x^2(\ 1 - \frac{1-e^{-x}}{F_0}\ )^{n-2}\ (\ 1 - \frac{1-e^{-\alpha x}}{G_0}\ )\ \frac{e^{-x}}{F_0}\ dx$$

$$+ (n-2)\iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-3} \left(1 - \frac{1-e^{-\alpha y}}{G_0} \right) \frac{e^{-y}}{F_0} \frac{e^{-x}}{F_0} \, dx \, dy$$

$$+ \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{F_0} \, dx \, dy$$

$$- \frac{e^{-x_0}}{F_0} \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-3} \left(1 - \frac{1-e^{-\alpha x}}{G_0} \right) \frac{e^{-x}}{F_0} \, dx$$

$$+ \frac{e^{-x_0}}{F_0} (n-3)\iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-4} \left(1 - \frac{1-e^{-\alpha y}}{G_0} \right) \frac{e^{-y}}{F_0} \frac{e^{-x}}{F_0} \, dx \, dy$$

$$+ \frac{e^{-x_0}}{F_0} \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-3} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{G_0} \, dx \, dy \Bigg]$$

Similarly I_2 and I_3 are simplified and substituted the values of I_1, I_2 and I_3 in (2.4.12). And using (1.6.2) and (1.6.4), it gives

$$\mu_{1:n} = -(n-1)\, \mu_{1:n}^{(2)} - \frac{e^{-x_0}}{F_0} (n-1)\, \mu_{1:n-1}^{(2)}$$

$$+ (n-1)\int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-2} \left(1 - \frac{1-e^{-\alpha x}}{G_0} \right) \frac{e^{-x}}{F_0} \, dx$$

$$(n-1) \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha x}}{G_0} \, dx + (n-2)\, \mu_{1,2:n}$$

$$+ \frac{e^{-x}}{F_0} (n-1)\, \mu_{1,2:n-1} + I_5 \qquad \ldots(2.4.13)$$

where

$$I_5 = (n-1)(n-2)\Bigg[\iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{G_0} \, dx \, dy$$

$$+ \frac{e^{-x}}{F_0} \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-3} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{G_0} \, dx \, dy \Bigg]$$

$$+ \quad (n-1)\,\alpha \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0} \right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha\, e^{-\alpha y}}{G_0}\, dx\, dy$$

after simplifying value of I_5 and put in (2.4.13),it reduce to

$$\mu_{1:n} = -\,(n-1)\,\mu^{(2)}_{1:n} - \frac{e^{-x}}{F_0}\,(n-1)\,\mu^{(2)}_{1:n-1} + (n-2+\alpha)\,\mu^{(2)}_{1,2:n}$$

$$+ \frac{e^{-x}}{F_0}\,(n-1)\,\mu^{(2)}_{1,2:n-1} + \frac{\alpha\, e^{-\alpha x_0}}{G_0}\,\nu_{1,2:n-1} + \frac{(1-\alpha)}{\alpha\, G_0}\,\nu_{1:n-1}$$

$$+ \left(1 - \frac{1}{G_0} \right) \nu^{(2)}_{1:n} - (n-1)\,(1-\alpha)\, I_7$$

$$I_7 = \int_0^{x_0} x\left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-2} \frac{e^{-x(\alpha+1)}}{\alpha\, G_0 F_0}\, dx$$

This can be written as

$$I_7 = \int_0^{x_0} x \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-1} \frac{e^{-\alpha x}}{\alpha G_0}$$

$$+ \frac{e^{-x_0}}{F_0} \int_0^{x_0} x \left(1 - \frac{1-e^{-x}}{F_0} \right)^{n-2} \frac{e^{-\alpha x}}{\alpha G_0}\, dx$$

Using (1.3.6) and (1.6.4) for truncated exponential model I_7 can be rewritten as

$$I_7 = \frac{1}{2\alpha} \left[\mu^{(2)}_{1:n} + \frac{e^{-\alpha x}}{G_0}\,\nu^{(2)}_{1:n-1} + \frac{e^{-x_0}}{F_0} \left\{ \mu^{(2)}_{1:n-1} + \frac{e^{-\alpha x_0}}{G_0}\,\nu^{(2)}_{1:n-1} \right\} \right]$$

Substituting the value of I_7 in (2.4.14) we get the required result.

The proof of ((2.4.11) is similar to the proof of (2.4.10).

COROLLARY 2.4.4: For samples containing a single outlier from an exponential distribution, and for $n = 3, 4, \ldots$,

$$\mu_{1,2:n} = \frac{1}{(n-2+\alpha)} \left[\mu_{1:n} + \frac{(n-1)(\alpha+1)}{2\alpha} \mu^{(2)}_{1:n} - \frac{(1-\alpha)}{\alpha} \nu_{1:n-1} \right] \quad \ldots(2.4.15)$$

and for $n = 2, 3, \ldots$

$$\mu_{n-1,n:n} = \mu_{n-1:n} + \frac{(n-1)(\alpha+1)}{2\alpha} \left(\mu^{(2)}_{n-1:n-1} - \mu^{(2)}_{n:n} \right) + \frac{(1-\alpha)}{\alpha} \nu_{n-1:n-1} + \nu^{(2)}_{n-1:n-1} \quad \ldots(2.4.16)$$

THEOREM 2.4.5 : (Shubha and Joshi, 1991)

For $r = 1, 2, \ldots\ldots, n-1$

For samples from exponential distribution, containing an outlier, we have

$$\mu_{r,r+1:n} = (n-r-1+\alpha)^{-1} \left[\frac{(n-r-1+\alpha)}{n-r} \mu_{r:n} + (n-r-1/2 + \alpha/2)\, \mu^{(2)}_{r:n} - \frac{(1-\alpha)}{2} \nu_{r:n-1} + \frac{(1-\alpha)}{n-r} \nu_{r:n-1} \right] \quad \ldots(2.4.17)$$

and for $s-r \geq 2$ and $1 \leq r < s \leq n$

$$\mu_{r,s:n} = \frac{1}{(n-s+\alpha)} \left[\mu_{1:n} + (n-s+1)\, \mu_{r,s-1:n} - \frac{(1-\alpha)}{2} \mu^{(2)}_{r:n} - (1-\alpha)\, \mu_{r:n} \sum_{j=0}^{s-r-1} \frac{1}{j+n-s+1} - \frac{(1-\alpha)}{2} \nu^{(2)}_{r:n-1} + \frac{(1-\alpha)}{(n-r)} \nu^{(2)}_{r:n-1} \right] \quad \ldots(2.4.18)$$

PROOF: The proof of (2.4.17) and (2.4.18) are analogous and hence we prove only (2.4.18). This technique is same as technique used by Joshi(1982), Thereby writing

$$\mu_{r:n} = E\,(X_{r:n}\, X^{0}_{s:n}) \quad \text{for} \quad s \geq r+2$$

Using (1.5.5) and (1.6.2) we have

$$\mu_{r:n} = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \iint_{W_1} x(1-e^{-x})^{r-2}\, e^{-x} (e^{-x} - e^{-y})^{s-r-1}$$

$$(1-e^{-\alpha x})e^{-(-n-s+1)y}\,dxdy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!}\iint_{w_1} x(1-e^{-x})^{r-1}$$

$$(e^{-x}-e^{-y})^{s-r-1} e^{-x}e^{-(n-s+1)\alpha}\,dxdy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}\Big[\alpha\iint_{w_1} x$$

$$(1-e^{-x})^{r-1}(e^{-x}-e^{-y})^{s-r-1} e^{-x}e^{-(n-s+\alpha)y}\,dx\,dy + \alpha\iint_{w_1} x(1-e^{-x})^{r-1}$$

$$(e^{-x}-e^{-y})^{s-r-1} e^{-\alpha x}e^{-(n-s+1)y}\,dx\,dy\Big] + \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!}$$

$$\iint_{w_1} x\,(1-e^{-x})^{r-1}(e^{-x}-e^{-y})^{s-r-2}(e^{-\alpha x}-e^{-\alpha y})\,e^{-x}e^{y(n-s+1)}\,dxdy$$

Where $w_1 = \Big\{ (x, y) : 0 < x < y < \infty \Big\}$

We write $\mu_{r:n} = I_1 + I_2 + I_3 + I_4 + I_5$ ...(2.4.19)

where for r=1, I_1 is zero and for s=n, I_2 is zero. First consider I_1

$$I_1 = \frac{(n-1)!}{(r-2)!(n-s)!(s-r-1)!}\int_0^\infty x\,(1-e^{-x})^{r-2}(1-e^{-\alpha x})e^{-x}\,I_0(x)\,dx$$

Where

$$I_0(x) = \int_x^\infty e^{-(n-s+1)y}(e^{-x}-e^{-y})^{s-r-1}\,dy$$

Integrating $I_0(x)$ treating 1 for Integration and $e^{-(n-s+1)y}(e^{-x}-e^{-y})^{s-r-1}$ for differentiation and substituting this value in I_1, which gives

$$I_1 = \frac{(n-1)!}{(r-2)!(n-s)!(s-r-1)!} \Big[(n-s+1) \iint\limits_{W_1} xy \Big\{ (1-e^{-x})^{r-2} e^{-x}$$

$$(1-e^{-\alpha x})(e^{-x}-e^{-y})^{s-r-1} e^{-(n-s+1)y} dxdy - (s-r-1)\iint\limits_{W_1} xy(1-e^{-x})^{r-2}$$

$$e^{-x} (1-e^{-\alpha x})(e^{-x}-e^{-y})^{s-r-2} e^{-(n-s+2)y} dxdy \Big]$$

similarly we write expression for I_2 , ..., I_5 and substituting these values in (2.4.19) and using (1.5.5) and (1.6.2), we get

$$\overline{\mu}_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)\Big[\frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!}$$

$$\iint\limits_{W_1} xy \, (1-e^{-x})^{r-2} (e^{-x}-e^{-y})^{s-r-1} e^{-x} e^{-(n-s+1)y} (1-e^{-\alpha x}) \, dx \, dy$$

$$+ \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!} \iint\limits_{W_1} xy \, (1-e^{-x})^{r-1} (e^{-x}-e^{-y})^{s-r-1} \alpha e^{-\alpha x}$$

$$e^{-(n-s+1)y} dx \, dy + \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \iint\limits_{W_1} xy \, (1-e^{-x})^{r-1} e^{-x}$$

$$(e^{-x}-e^{-y})^{s-r-2} e^{-\alpha x} e^{-(n-s+1)y} dx \, dy .$$

After substitutions and simplification and using (1.6.4) for $k = 1, 2$, we obtain

$$\overline{\mu}_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)\Big[\mu^{(2)}_{r:n} + \mu_{r:n}$$

$$\sum_{j=0}^{s-r-1} \frac{1}{j+n-s+1} - \binom{n-1}{r-1} \int_0^\infty x \, (1-e^{-x})^{r-1} e^{-(n-r+\alpha)x} \, dx \Big]$$

Now again using (1.3.6) and (1.6.4) for $k = 2$ with (1.3.5) it reduces to

$$\mu_{r:n} = (n-s+\alpha)\,\mu_{r,s:n} - (n-s+1)\,\mu_{r,s-1:n} - (\alpha-1)\Bigg[\,\mu^{(2)}_{r:n} + \mu_{r:n}$$

$$\sum_{j=0}^{s-r-1} \frac{1}{j+n-s+1} - \frac{1}{2}\left\{\mu^{(2)}_{r:n} - \nu^{(2)}_{r:n-1} + \frac{2}{n-r}\,\nu_{r:n-1}\right\}\Bigg]$$

After simplification it gives the result.

COROLLARY 2.4.5: *For a random sample from* exponential distribution we have

$$\nu_{r,s:n} - \nu_{r,s-1:n} = \frac{1}{n-s+1}\,\nu_{r:n}$$

PROOF: Substituting $\alpha = 1$ (2.4 .18) , we get Corollary (2.4.5) .

2.5 IDENTITIES FOR PRODUCT MOMENTS (Balakrishnan, 1988)

For arbitrary continuous distributions F(x) and G(x), it is known from David et al.(1977), is that

$$\sum_{r=1}^{n} \mu^{(k)}_{r:n} = (n-1)\,E(x^k) + E(y^k) \qquad \ldots(2.5.1)$$

and

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \sigma_{r,s:n} = (n-1)\,\mathrm{var}(x) + \mathrm{var}(y) \qquad \ldots(2.5.2)$$

These relations are often used for checking the computations of means, variances and covariances of order statistics from a single outlier model. We now derive new identities involving linear combinations of product moments and covariences. There are quite simple and more effective for checking the calculations of covariences.

THEOREM 2.5.1: (Balakrishnan, 1988)

For arbitrary continuous distributions F(x) and G(x) , we have for $1 \le k \le n-1$

$$\sum_{s=2}^{n-k+1} \binom{n-s}{k-1} \mu_{1,s:n} + \sum_{s=2}^{k+1} \binom{n-s}{n-k-1} \mu_{1,s:n} = \binom{n-1}{k} \nu_{1:k}\ \mu_{1:1-k}$$

$$+ \binom{n-1}{k-1} \nu_{1:k}\ \mu_{1:k} \qquad \ldots(2.5.3)$$

PROOF: First, we consider the expression for $\sum_{s=2}^{n-k+1} \binom{n-s}{k-1} \mu_{1,s:n}$ from (1.6.2). Upon interchanging the summation and the integral sign and then simplifying, we obtain

$$\sum_{s=2}^{n-k+1} \binom{n-s}{k-1} \mu_{1,s:n} = \iint_{w_1} \overline{xy}\ H_{k,n}(x,y)\ dx\ dy \qquad \ldots(2.5.4)$$

where

$$H_{k,n}(x,y) = \frac{(n-1)!}{(k-1)!(n-k-1)!} \left\{1 - F(x)\right\}^{n-k-1} \left\{1 - F(y)\right\}^{k-1} g(x)\ f(y)$$

$$\frac{(n-1)!}{(k-1)!(n-k-1)!} \left\{1 - F(x)\right\}^{n-k-1} \left\{1 - F(y)\right\}^{k-1} g(x)\ f(x)$$

$$+ \frac{(n-1)!}{(k-1)!(n-k-2)!} \left\{1 - F(x)\right\}^{n-k-2} \left\{1 - F(y)\right\}^{k-1} \left\{1 - G(x)\right\} f(x)\ f(y)$$

$$+ \frac{(n-1)!}{(k-2)!(n-k-1)!} \left\{1 - F(x)\right\}^{n-k-1} \left\{1 - F(y)\right\}^{k-2} \left\{1 - G(x)\right\} f(x)\ f(y)$$

$$\ldots(2.5.5)$$

Next we consider the expression for $\sum_{s=2}^{k+1} \binom{n-s}{n-k-1} \mu_{1,s:n}$ from (2.4.1). Upon enter changing the summation and the integral signs and then simplifying as before, we obtain

$$\sum_{s=2}^{k+1} \binom{n-s}{n-k-1} \mu_{1,s:n} = \iint_{w_2} xy\ H_{k,n}(x,y)\ dx\ dy \qquad \ldots(2,.5.6)$$

where $H_{k,n}(x,y)$ is defined in (2.5.5). Finally, upon adding

(2.5.4) and (2.5.6), noting that $w_1 \cup w_2 = R^2$, and then simplifyingthe resulting expression using (1.6.1)and (1.3.1), we derive the identity in (2.5.3).

REMARK 2.5.1: Here, it is important to note that (2.5.3) contains product moments $\mu_{1,s:n}$,$2 \le s \le n$, and first order single moments only and there are only (n/2) distinct equations since (2.5.3) for k is same as for n−k . Thus, for even values of n, there are only n/2 in n−1 product moments and so we need to know exactly (n−2)/2 of them. Similarly, for odd values of n, we only need to know (n−1)/2 of these product moments. It is just as given in theorem (2.4.3) since the product moments $\mu_{1,s:n}$, $2 \le s \le n$, along with relation (2.4.3) is also sufficient for the evaluation of all product moments.

THEOREM 2.5.2: (Balakrishnan, 1988)

For arbitrary continuous distributions F(x) and G(x), we have

$$\sum_{r=1}^{n-1} \mu_{r,r+1:n} + \sum_{j=2}^{n} \binom{n-1}{j-1} \mu_{1,j:j} = \sum_{j=1}^{n-1} \left\{ \binom{n-1}{j-1} \nu_{1:n-j}\, \mu_{j:j} + \binom{n-1}{j} \nu_{j:j}\, \mu_{1:n-j} \right\} - \sum_{j=2}^{n-1} \binom{n-1}{j} \nu_{1,j:j} \qquad \ldots(2.5.7)$$

PROOF: Consider the sum of integrals

$$I = \sum_{r=1}^{n-1} \iint_{w_1} xy\, h_{r,r+1:n}(x,y)\, dxdy + \sum_{r=1}^{n-1} \iint_{w_2} xy\, h_{r,r+1:n}(x,y)\, dxdy$$

$$= \sum_{r=1}^{n-1} \mu_{r,r+1:n} + J \qquad \ldots(2.5.8)$$

where

$$J = \sum_{r=1}^{n-1} \iint_{w_2} xy\, h_{r,r+1:n}(x,y)\, dx\, dy$$

Now upon interchanging the summation and the integral signs, and using the binomial identity that

$$\sum_{j=0}^{m} \binom{m}{j} \left\{ F(x) \right\}^{j} \left\{1 - F(y)\right\}^{m-j} = \left\{ 1 + F(x) - F(y) \right\}^{m},$$

expanding the term $\left\{ 1 + F(x) - F(y) \right\}^{m}$ binomially in powers of $\left\{ F(x) - F(y) \right\}$, and then simplifying the resulting expression using (2.4.1) and (2.4.2), we obtain

$$J = \sum_{j=2}^{n} \binom{n-1}{j-1} \mu_{1,j:j} + \sum_{j=2}^{n-1} \binom{n-1}{j} \nu_{1,j:j}$$

which, when substituted in (2.5.8) , yields

$$I = \sum_{r=1}^{n-1} \mu_{r,r+1:n} + \sum_{j=2}^{n} \binom{n-1}{j-1} \mu_{1,j:j} + \sum_{j=2}^{n-1} \binom{n-1}{j} \nu_{1,j:j}$$

Further , we can write

$$I = \sum_{r=1}^{n-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \; h_{r,r+1:n}(x,y) \; dx \; dy$$

Now writing each term as a product of two single integrals, one involving X alone and other involving Y alone and then simplifying the resulting expression using (1.3.1) and (1.6.1), we also obtain

$$I = \sum_{r=1}^{n-1} \left\{ \binom{n-1}{j-1} \nu_{j:n-j} \, \mu_{j:j} + \binom{n-1}{j} \nu_{j:j} \, \mu_{1:n-j} \right\}$$

Thus Identity (2.5.7) follows upon equating above two expression for I.

THEOREM 2.5.3: (Balakrishnan, 1988)

For arbitrary continuous distribution F(x) and G(x), we have for $1 \le r \le n-1$

$$\sum_{s=r+1}^{n} \mu_{r,s:n} + \sum_{i=1}^{r} \mu_{1,r+1:n} = (n-1) \, \mu_{r:n-1} \, \nu_{1:1} + \nu_{r:n-i} \, \mu_{i:i} \qquad \ldots(2.5.9)$$

PROOF: Let us consider the expression for $\sum_{s=r+1}^{n} \mu_{r,s:n}$ from (1.6.2). Now upon interchanging the summation and integral signs and then simplifying, we obtain

$$\sum_{s=r+1}^{n} \mu_{r,s:n} = \iint_{w_1} xy\, h_{r,s,n}(x,y)\, dx\, dy \qquad \ldots(2.5.10)$$

where $h_{r,s,n}(x,y)$ is the pdf of r^{th} and s^{th} order statistics in the presence of single outlier.

Next, we consider the expression for $\sum_{i=1}^{r} \mu_{i,r+1:n}$ from (2.4.2).

Upon interchanging the summation and the integral signs and then simplifying as before, we obtain

$$\sum_{i=1}^{r} \mu_{i,r+1:n} = \iint_{w_2} xy\, h_{r,s,n}(x,y)\, dxdy \qquad \ldots(2.5.11)$$

Finally, upon adding (2.5.10) and (2.5.11), noting that $w_1 \cup w_2 = R^2$, and then simplifying the resulting expression using (1.3.1) and (1.6.1) we get the result.

COROLLARY 2.5.1:

For arbitrary continuous distributions $F(x)$ and $G(x)$, we have for $1 \le r \le n-1$

$$\sum_{s=r+1}^{n} \sigma_{r,s:n} + \sum_{i=1}^{r} \sigma_{i,r+1:n} = (r\, \nu_{1:1} - \sum_{i=1}^{r} \mu_{i:n})(\mu_{r+1:n} - \mu_{r:n})$$
$$- (\mu_{1:1} - \nu_{1:1})(\mu_{r:n} - \nu_{r:n-1}) \qquad \ldots(2.5.12)$$

PROOF: The above result follows directly upon using relation (2.2.1) and identity (2.5.1) in theorem (2.5.3).

Both (2.5.9) and (2.5.12) give extremely simple and useful identities for checking the calculations of product moments and covariences of order statistics from a sample of size n comprising one outlier. In practical, setting r= 1 and r = n-1 in (2.5.12) we get the identities

$$2\,\sigma_{1,2:n} + \sum_{s=3}^{n} \sigma_{1,s:n} = (\nu_{1:1} - \mu_{1:n})(\mu_{2:n} - \mu_{1:n})$$
$$- (\mu_{1:1} - \nu_{1:1})(\mu_{1:n} - \nu_{1:n-1})$$

and

$$2\,\sigma_{n-1,n:n} + \sum_{s=1}^{n-2} \sigma_{s,n:n} = (\mu_{n:n} - \mu_{1:1})(\mu_{n:n} - \nu_{n-1:n})$$
$$- (\mu_{1:1} - \nu_{1:1})(\mu_{n-1:n} - \nu_{n-1:n-1})$$
$$= (\mu_{n:n} - \mu_{1:1})(\mu_{n:n} - \nu_{n-1:n-1})$$
$$- (\mu_{n:n} - \nu_{1:1})(\mu_{n-1:n} - \nu_{n-1:n-1})$$

respectively.

2.6 THEOREMS RELATED WITH SYMMETRIC DISTRIBUTIONS

Let us consider the case when the density functions f(x) and g(x) both are symmetric about zero. It is easy to see that

$$\mu_{r:n}^{(k)} = (-1)^{k}\,\mu_{n-r+1:n}^{(k)}$$

and

$$\mu_{r,s:n} = \mu_{n-s+1,n-r+1:n}\,.$$

From theorem (2.2.1), we obtain

$$\mu_{n/2:n}^{(2)} = \left\{ (n-1)\,\mu_{n/2:n}^{(2)} + \nu_{n/2:n-1}^{(2)} \right\} / \; n \qquad \ldots(2.6.1)$$

for even value of n ; and

$$\mu_{(n+1)/2:n}^{(2)} = 0 \qquad \ldots(2.6.2)$$

for odd value of n. Moreover, we obtain from theorem (2.4.2) that, for even valve of n,

$$
\begin{aligned}
2\,\mu_{r,s:n} &= 2\,\mu_{n-s+1,n-r+1:n} \\
&= 2\,\nu_{r,s:n} + \sum_{k=1}^{r-1} (-1)^k \binom{n-1}{k} \left\{ \mu_{n-s+1,n-r+1:n-k} - \nu_{n-s+1,n-r+1:n-k} \right\} \\
&\quad + \sum_{j=1}^{n-s} \sum_{k=0}^{r-1} (-1)^{j+k-1} \binom{n-1}{j} \binom{n-1-j}{k} \\
&\quad \left\{ \mu_{n-s-j+1,n-r-j+1:n-j-k} - \nu_{n-s-j+1,n-r-j+1:n-j-k} \right\} \\
&\quad + \frac{1}{n} \sum_{j=1}^{s-r} (-1)^{s-r-j} \left[\binom{n}{s-j} \right] \binom{n}{s-j} \binom{s-j-1}{r-1} \\
&\quad \left\{ (s-j)\,\nu_{j:n-s-j} \left(\mu_{s-j:s-j} - \nu_{s-j:s-j} \right) \right. \\
&\quad \left. + (n-s+j)\,\nu_{s-j:s-j} \left(\mu_{j:n-s+j} - \nu_{j:n-s+j} \right) \right\}
\end{aligned} \qquad \ldots(2.6.3)
$$

Now with the use of (2.6.1)—(2.6.3), we then have the following theorem which generalize Joshi (1971) results.

THEOREM 2.6.1: (Balakrishnan, 1988)

In order to find the first, second and product moments of order statistics in a sample of size n from a single outlier model with densities f(x) and g(x) both symmetric about zero, given these moments for all sample sizes less than n, one has to evaluate at most one single moment if n is even and one single moment and (n−1)/2 product moments if n is odd.

furthermore, by setting r = n−1 and s = n in (2.4.3) and using the fact that $\mu_{1:1} = \nu_{1:1} = 0$, we get

$$
2\,\mu_{1,n:n} = 2\,\nu_{1,2:n} + \sum_{k=1}^{n-2} (-1)^k \binom{n-1}{k} \left\{ \mu_{1,2:n-k} - \nu_{1,2:n-k} \right\}
$$

which is established by using the result of Joshi(1971).

$$2\,\nu_{1,2:n} = \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} \nu_{1,2:n-k} ,$$

yields the recurrence relation

$$2\,\mu_{1,2:n} = \sum_{k=1}^{n-2} (-1)^k \left\{ \binom{n-1}{k} \mu_{1,2:n-k} + \binom{n-1}{k-1} \nu_{1,2:n-k} \right\} \qquad \ldots(2.6.4)$$

for even value of n.

Chapter III

RECURRENCE RELATIONS FOR INDEPENDENT NONIDENTICALLY DISTRIBUTE RANDOM VARIABLES

3.1 INTRODUCTION

Recurrence relations for order statistics from n independent and nonidentically distributed random variables given by Balakrishnan (1988b). Bapat and Beg (1989 a & b) established recurrence relations for independent nonidentically distributed exponential random variables. Recurrence relations for product of moments and product moments and identities for product moments are given by Balasubramanian and Beg(1991). Bapat and Beg(Preprint.95) established two simple identities and some recurrence relations involving order statistics from a sample of size n containing one or more than one outlioers. Balakrishnan(1994b) derived several new relations for single and product moments in exponentially distributed random variables and generalize these results for multiple outlier models. Recurrence relations for single and product moments in right truncated exponential distribution and generalize these results for p-outlier model are derived by Balakrishnan(1994).

3.2 RECURRENCE RELATIONS FOR DISTRIBUTION FUNCTION AND PROBABILITY DENSITY FUNCTION

Let $H_{r:n}(x)$ denote the ditribution funtion and $h_{r:n}(x)$ denote the density function of $X_{r:n}$, $1 \le r \le n$. Let $h_{r,s:n}(x,y)$ denotethe joint density function of $X_{r,n}$ and $X_{s,n}$. and $N = \{1, 2, \ldots, n\}$. If $S \subset N$ then S' will denote the complement of S in N while $|S|$ will denote the cardinality of S. Let $X^{s}_{r:|S|}$ denote the r^{th} order statistics corresponding to X_i, $i \in s$. Suppose $H^{s}_{r:|S|}$ and $h^{s}_{r:|S|}$ denote the ditribution function and the density function of $X^{s}_{r:|S|}$ respectively. For convenience, for fixed x, F will denote the column vectors $(F_1(x), F_2(x), \ldots, F_n(x))'$ and 1 the column vector of all ones. We will denote, $A(i,j)$ the matrix obtained by deleting i rows and j columns of A and $A[S|.)$, the matrix obtain by taking rows whose indices are i n S. Here we review generalized results of Joshi(1973) and Balakrishnan(1987).

THEOREM 3.2.1: (Bapat and Beg, Pre-Printed, 1995)

For arbitrary distribution $F, F, \ldots, F$ and $n \ge 2$,

$$a = \sum_{r=1}^{n} \frac{1}{(n-r+1)} H_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r\binom{n}{r}} \sum_{|S|=r} H^{s}_{r:r}(x) \qquad \ldots(3.2.1)$$

$$b = \sum_{r=1}^{n} \frac{1}{r} H_{r:n}(x) = \sum_{r=i}^{n} \frac{1}{r\binom{n}{r}} \sum_{|S|=r} H^{s}_{1:r}(x) \qquad \ldots(3.2.2)$$

PROOF: (a) The distribution function of $X_{r:n}$ ($1 \le r \le n$) is given by Bapat and Beg,(1988).

$$H_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \operatorname{Per} [\, \underset{i}{\underline{F}} \,,\, \underset{n-i}{\underline{1-F}} \,]$$

$$= \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{t=0}^{n-i} \binom{n-i}{t} \text{Per} \, [\underbrace{F}_{i} , \underbrace{-F}_{t} , \underbrace{1}_{n-i-t}]$$

$$= \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{t=0}^{n-i} \binom{n-i}{t} (-1)^{t} \, \text{Per} \, [\underbrace{F}_{i+t} , \underbrace{1}_{n-i-t}]$$

$$= \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{t=0}^{n-i} \binom{n-i}{t} (-1)^{t} \sum_{|S|=i+t} (n-i-t)! \text{Per}[\underbrace{F}_{i+t}][S|.)$$

$$= \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{t=0}^{n-i} \binom{n-i}{t} (-1)^{t} \sum_{|S|=i+t} (n-i-t)! \; (i+t)! \; H^{S}_{i+t:i+t}(x)$$

Therefore

$$\sum_{r=1}^{n} \frac{1}{(n-r+1)} H_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{(n-r+1)} \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{t=0}^{n-i} \binom{n-i}{t} (-1)^{t} \sum_{|S|=i+t} (n-i-t)! \; (i+t)! \; H^{S}_{i+t:i+t}(x)$$

Writing $z = i+t$, we get

$$\sum_{r=1}^{n} \frac{1}{(n-r+1)} H_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{(n-r+1)} \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \sum_{z=i}^{n} \binom{n-i}{z-i} (-1)^{z-i} \sum_{|S|=z} (n-z)! \; (z)! \; H^{S}_{z:z}(x)$$

$$= \sum_{r=1}^{n} \sum_{i=r}^{n} \sum_{z=i}^{n} \frac{(-1)^{z-i}}{(n-r+1)} \binom{z}{i} \sum_{|S|=z} H^{S}_{z:z}(x)$$

$$= \sum_{r=1}^{n} \sum_{z=r}^{n} \sum_{i=r}^{n} \frac{(-1)^{z-i}}{(n-r+1)} \binom{z}{i} \sum_{|S|=z} H^{S}_{z:z}(x)$$

$$= \sum_{z=1}^{n} \left\{ \sum_{r=1}^{z} \sum_{i=r}^{z} \frac{(-1)^{z-i}}{(n-r+1)} \binom{z}{i} \right\} \sum_{|S|=z} H^{S}_{z:z}(x)$$

since

$$\sum_{i=r}^{z} (-1)^{z-i} \binom{z}{i} = (-1)^{z-r} \binom{z-1}{z-r}$$

the expression in the braces become

$$\sum_{i=r}^{z} \binom{z-1}{z-r} \frac{(-1)^{z-r}}{(n-r+1)}$$

Moreover, evaluating the integral

$$\int_0^1 \sum_{r=1}^{z} \binom{z-1}{z-r} (-1)^{z-r} \, u^{n-r} \, du$$

in two different ways it can be shown that

$$\sum_{r=1}^{z} \binom{z-1}{z-r} \frac{(-1)^{z-r}}{(n-r+1)} = \frac{1}{z\binom{n}{z}}$$

Thus the result follows.

The proof of (b) is similar. It also holds for pdf's, characteristic functions and raw moments.

Consider the set up in which variables X_1, X_2, ..., X_{n-1} are identically distributed with dF F and X_n is an outlier with df F_n then the relations (a) and (b) of Theorem (3.2.1) yield

$$\sum_{r=1}^{n} \frac{1}{(n-r+1)} H_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r\binom{n}{r}} \left\{\binom{n-1}{r-1} H_{r:r}(x) + \binom{n-1}{r} F_{r:r}(x)\right\}$$

$$= \frac{1}{n} \sum_{r=1}^{n} H_{r:r}(x) + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] F_{r:r}(x) \qquad \ldots(3.2.3)$$

$$\sum_{r=1}^{n} \frac{1}{n} H_{r:n}(x) = \frac{1}{n} \sum_{r=1}^{n} H_{1:r}(x) + \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{n} \right] F_{1:r}(x) \qquad \ldots(3.2.4)$$

which have been established for absolutely continuous distributions by Balakrishnan (1987).

Bapat and Beg(1989a) generalized the result of Krishnaian and Rizvi(1966), which are follows.

THEOREM 3.2.2: (Bapat and Beg, 1989a)

For $1 \le r \le n$,

$$\text{(a)}\quad h_{r:n}(x) = \binom{n}{r} \sum_{s=0}^{i} (-1)^s \left(\frac{r}{r-i}\right)\binom{i}{s}$$

$$\left[\frac{(r-i)!(i-s)!(n-r+s)}{n!}\right] \sum_{|S|=n-i+s} h^{S}_{r-i:n-i+s}(x), \quad 0 \le i \le r-1 \qquad \ldots(3.2.5)$$

$$\text{(b)}\quad h_{r:n}(x) = \binom{n}{r} \sum_{s=0}^{j} (-1)^s \left(\frac{r}{r+s}\right)\binom{j}{s}\left[\frac{(r+s)!(j-s)!(n-r-j)!}{n!}\right]$$

$$\sum_{|S|=n-j+s} h^{S}_{r+s:n-j+s}(x), \quad 0 \le j \le n-r \qquad \ldots(3.2.6)$$

PROOF: The density function of $X_{r:n}$, $1 \le r \le n$, is given as

$$h_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \operatorname{Per}\left[\underset{1}{\underline{f}}, \underset{r-1}{\underline{F}}, \underset{n-r}{\underline{1-F}}\right]$$

$$= \frac{1}{(r-1)!(n-r)!} \operatorname{Per}\left[\underset{1}{\underline{f}}, \underset{i}{\underline{F}}, \underset{r-i-1}{\underline{F}}, \underset{n-r}{\underline{1-F}}\right], \quad 0 \le i \le r-1$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{i} \binom{i}{t} \operatorname{Per}\left[\underset{1}{\underline{f}}, \underset{t}{\underline{1}}, \underset{i-t}{\underline{-(1-F)}}, \underset{r-i-1}{\underline{F}}, \underset{n-r}{\underline{1-F}}\right]$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{i} \binom{i}{t} (-1)^{i-t} \operatorname{Per}\left[\underset{1}{\underline{f}}, \underset{t}{\underline{1}}, \underset{r-i-1}{\underline{F}}, \underset{n-i-r-t}{\underline{1-F}}\right]$$

$$= \frac{1}{(r-1)!(n-r)!} \binom{i}{t}(-1)^{i-t} \sum_{|S|=n-t} t!\ \mathrm{Per}\left[\ \underset{1}{f}\ ,\ \underset{t}{1}\ ,\ \underset{n+i-r-t}{1-F}\ \right]$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{i} \binom{i}{t}(-1)^{i-t} \sum_{|S|=n-t} t!\ \mathrm{Per}\left[\ \underset{1}{f}\ ,\ \underset{r-i-1}{F}\ ,\ \underset{n+i-r-t}{1-F}\ \right][S|.)$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{i} \binom{i}{t}(-1)^{i-t} \sum_{|S|=n-t} t!(r-i-1)!(n+i-r-t)!\,h^{S}_{r-i:n-t}(x)$$

writing $s = i-t$, we get

$$h_{r:n}(x) = \binom{n}{r} \sum_{s=0}^{i} (-1)^{s} \binom{r}{r-i} \binom{i}{s} \left[\frac{(r-i)!(i-s)!(n-r+s)!}{n!} \right]$$

$$\sum_{|S|=n-i+s} h^{S}_{r-i:n-i+s}(x)\ ,$$

also for $1 \le r \le n$ and $0 \le j \le n-r$

$$h_{r:n}(x) = \frac{1}{(r-1)!(n-r)!}\ \mathrm{Per}\left[\ \underset{1}{f}\ ,\ \underset{r-1}{F}\ ,\ \underset{j}{1-F}\ ,\ \underset{n-r-j}{1-F}\ \right]$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{j} \binom{j}{t}(-1)^{j-t}\ \mathrm{Per}\left[\ \underset{1}{f}\ ,\ \underset{r+j-t-1}{F}\ ,\ \underset{n-r-j}{1-F}\ ,\ \underset{t}{1}\ \right]$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_{t=0}^{j} \binom{j}{t} (-1)^{j-t}$$

$$\sum_{|S|=n-t} t!(r+j-t-1)!(n-r-j)!\,h^{S}_{r+j-t:n-t}(x)$$

writing $s = j-t$, we get

$$h_{r:n}(x) = \binom{n}{r} \sum_{s=0}^{j} (-1)^{s} \binom{r}{r+s} \binom{j}{s} \left[\frac{(r+s)!(j-s)!(n-r-j)!}{n!} \right]$$

$$\sum_{|S|=n-j+s} h^{S}_{r+s:n-j+s}(x)\ ,$$

Thats prove the theorem.

In the above Theorem (3.2.2) If we multiply both sides by $g(x)$ and integrate with respect to x then we obtain recurrence relations for moments which generalize the results of Krishnaiah and Rizvi (1966) for I.I.d case.

In particular, for $i = j = 1$, Theorem (3.2.2) yields

$$(n-r)\ h_{r:n}(x) + r\ h_{r+1:n}(x) = \sum_{|S|=n-1} h^{S}_{r:n-1}(x) \qquad \ldots(3.2.7)$$

which is Theorm 5.2 of Bapat and Beg (1988). If we consider variables $X_1, X_2, \ldots, X_{n-1}$ are distributed with df F and pdf F and X_n is an outlier with df F_n and pdf f_n, then

$$(n-r)\ h_{r:n}(x) + r\ h_{r+1:n}(x) = \binom{n-1}{n-2} h_{r:n-1}(x) + \binom{n-1}{n-1} f_{r:n-1}(x)$$

$$= (n-1)\ h_{r:n-1}(x) + f_{r:n-1}(x) \qquad \ldots(3.2.8)$$

THEOREM 3.2.3 : (Bapat and Beg, 1989a)

For $1 \le r \le s \le n$,

$$(r-1)\ h_{r,s:n}(x,y) + (s-r)\ h_{r-1,s:n}(x,y) + (n-s+1)\ h_{r-1,s-1:n}(x,y)$$

$$= \sum_{|S|=n-1} h^{S}_{r-1,s-1:n-1}(x,y)\ ,\ x < y \qquad \ldots(3.2.9)$$

PROOF : The joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by Vaughan & Venables, (1972) as

$$h_{r,s:n}(x,y) = \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-1}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big],\ x \le y$$

$$= \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big]$$

$$= \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{1-F(x)-(1-F(y))},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big]$$

$$= \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\Big\{ \operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)},\ \underset{1}{1}\Big]$$

$$- \operatorname{Per}\Big[\underset{r}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big]$$

$$- \operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s+1}{1-F(y)}\Big]\Big\}$$

$$= \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\Big\{ \sum_{|S|=n-1} \operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big][S|.)$$

$$- \operatorname{Per}\Big[\underset{r}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s}{1-F(y)}\Big]$$

$$- \operatorname{Per}\Big[\underset{r-1}{F(x)},\ \underset{1}{f(x)},\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)},\ \underset{n-s+1}{1-F(y)}\Big]\Big\}$$

since

$$h^{s}_{r,s-1:n-1}(x,y) = \frac{1}{(r-1)!\ (s-r-1)!\ (n-s)!}$$

$$\mathrm{Per}\ [\ \underset{r-1}{F(x)}\ ,\ \underset{1}{f(x)}\ ,\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)}\ ,\ \underset{n-s}{1-F(y)}\ ]\ [S|.)$$

$$h_{r+1,s:n}(x,y) = \frac{1}{(r)!\ (s-r-2)!\ (n-s)!}$$

$$\mathrm{Per}\ [\ \underset{r}{F(x)}\ ,\ \underset{1}{f(x)}\ ,\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)}\ ,\ \underset{n-s}{1-F(y)}\ ]$$

and

$$h_{r,s-1:n}(x,y) = \frac{1}{(r-1)!\ (s-r-2)!(n-s+1)}$$

$$\mathrm{Per}\ [\ \underset{r-1}{F(x)}\ ,\ \underset{1}{f(x)}\ ,\ \underset{s-r-2}{F(y)-F(x)},\ \underset{1}{f(y)}\ ,\ \underset{n-s+1}{1-F(y)}\ ]$$

the result follows by making a simple rearrangement of the terms and replacing r by $r-1$.

If we multipl y in Theorem(3.2.3) both sides by $g(x,y)$ and integrate with respect to x and y then we obtain a recurrence relation for product moments which generalize the result of Govindarajalu(1963) for i.i.d. case.

let we consider the sample of n independent absoluely continuous random variables X :($i=1, 2, \ldots, n-1$), where X_i has dF F and pdf f and X_n has dF F_n and pdf f_n, thereby from Theorem(3.2.2), we get,

$$(r-1)h_{r,s:n}(x,y) + (s-r)h_{r-1,s:n}(x,y) + (n-s+1)h_{r-1,s-1:n}(x,y)$$
$$= f_{r-1,s-1:n-1}(x,y) + (n-1)h_{r-1,s-1:n-1}(x,y) \quad , \quad x < y\ . \qquad \ldots(3.2.10)$$

If we use $h_{r:n-m}^{[i_1,\ldots,i_m]}(x)$, $1 \le r \le n-m$ to denote the density function of r^{th} order statistics in a sample of size $n-m$ obtained by dropping $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$ from the original set of n variables then we have following relations derived by Balakrishnan (1988b).

THEOREM 3.2.4: (Balakrishnan, 1988b)

For $1 \le r \le n-1$

$$r\, h_{r+1:n}(x) + (n-r)\, h_{r:n}(x) = \sum_{i=1}^{n} h^{i}_{r:n-1}(x) \qquad \ldots(3.2.11)$$

PROOF: First consider the permanent expression of $r\, h_{r+1:n}(x)$ from (1.9.1) and expanding ~~this~~ expression by its first row, we get

$$r\, h_{r+1:n}(x) = \sum_{i=1}^{n} F_i(x)\, h^{[i]}_{r:n-1}(x) \qquad \ldots(3.2.12)$$

further we consider the expression of $(n-r)\, h_{r:n}(x)$ from (1.9.1) and expanding the it by its last row , we get

$$(n-r)\, h_{r:n}(x) = \sum_{i=1}^{n} (1 - F_i(x))\, h^{[i]}_{r:n-1}(x) \qquad \ldots(3.2.13)$$

Now adding (3.2.13) and (3.2.12) which follows the Theorem(3.2.11).

let us denote

$$S_{1:n-m}(x) = \sum_{1\le i_1<i_2,\ldots,< i_m \le n} h^{[i_1, \ldots, i_m]}_{1:n-m}(x)$$

and

$$S_{n-m:n-m}(x) = \sum_{1\le i_1<i_2,\ldots,< i_m \le n} h^{[i_1, \ldots, i_m]}_{n-m:n-m}(x)$$

with $S_{1:n}(x) = h_{1:n}(x)$ and $S_{n:n}(x) = h_{n:n}(x)$.

Then by application of Theorem (3.2.1), We directly obtain the following relations.

THEOREM 3.2.5 : (Balakrishnan, 1988b)

For $1 \leq r \leq n-1$,

$$h_{r:n}(x) = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} S_{j:j}(x) \qquad \ldots(3.2.14)$$

THEOREM 3.2.6 : (Balakrishnan, 1988b)

For $2 \leq r \leq n$,

$$h_{r:n}(x) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} S_{1:j}(x) \qquad \ldots(3.2.15)$$

REMARK 3.2.1: For the case X_i's are identically distributed , it is easy to see

$$S_{j:j}(x) = \sum_{1\leq i_1<i_2,\ldots,<i_{n-j}\leq n} h_{j:j}^{[i_1,\ \ldots,\ i_{n-j}]}(x) = \binom{n}{j} h_{j:j}(x)$$

$$S_{1:j}(x) = \sum_{1\leq i_1<i_2,\ldots,<i_{n-j}\leq n} h_{1:j}^{[i_1,\ \ldots,\ i_{n-j}]}(x) = \binom{n}{j} h_{1:j}(x)$$

It terms of moments relations (3.2.14) and (3.2.15) are written as

$$\mu_{r:n}^{(k)} = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \binom{n}{j} \mu_{j:j}^{(k)} , \quad 1 \leq r \leq n-1 \qquad \ldots(3.2.16)$$

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \binom{n}{j} \mu_{1:j}^{(k)} , \quad 2 \leq r \leq n-1 \qquad \ldots(3.2.17)$$

Above two relations are quite well known and are due to Srikantan(1962).

Let us suppose $h_{r,s:n-1}^{[i]}(x,y)$, $1 \leq r < s \leq n-1$, denote the joint density function of the r^{th} and s^{th} order statistics in a sample of size n-1 obtained by dropping X_i from the original set of n variables , we have following recurrence relations.

THEOREM 3.2.7: (Balakrishnan, 1988b)

For $2 \le r < s \le n$,

$$(r-1)\, h_{r,s:n}(x,y) + (s-r)\, h_{r-1,s:n}(x,y) + (n-s+1)\, h_{r-1,s-1:n}(x,y)$$

$$= \sum_{i=1}^{n} h^{[i]}_{r-1,s-1:n-1}(x,y) \qquad \ldots(3.2.18)$$

PROOF: Expand the permanent expression of joint density function of its first r^{th} and last row respetively then we get

$$(r-1)\, h_{r,s:n}(x,y) = \sum_{i=1}^{n} F_i(x)\, h^{[i]}_{r-1,s-1:n-1}(x,y), \qquad \ldots(3.2.19)$$

$$(s-r)\, h_{r-1,s:n}(x,y) = \sum_{i=1}^{n} (\, F_i(y) - F_i(x))\, h^{[i]}_{r-1,s-1:n-1}(x,y), \qquad \ldots(3.2.20)$$

$$(n-s+1)\, h_{r-1,s-1:n}(x,y) = \sum_{i=1}^{n} (\, 1 - F_i(y))\, h^{[i]}_{r-1,s-1:n-1}(x,y), \qquad \ldots(3.2.21)$$

After adding (3.2.19), (3.2.20) and (3.2.21), we get relation of Theorem (3.2.7).

REMARK 3.2.2:

For the p-outlier model, that is $F_1 = F_2 = \ldots = F_{n-p} = F$ and $F_{n-p+1} = \ldots = F_n = G$, relations (3.2.11) and (3.2.18), yields the following results respectively.

$$r\, h_{r+1:n}(x) + (n-r)\, h_{r:n}(x) = (n-p)\, h^{[F]}_{r:n-1}(x) + p\, h^{[G]}_{r:n-1}(x)$$

and

$$(r-1)\, h_{r,s:n}(x,y) + (s-r)\, h_{r-1,s:n}(x,y) + (n-s+1)\, h_{r-1,s-1:n}(x,y)$$

$$= (n-p)\, h^{[F]}_{r-1,s-1:n-1}(x,y) \;+\; p\, h^{[G]}_{r-1,s-1:n-1}(x,y)$$

where $h^{[F]}_{r:n-1}(x)$ and $h^{[G]}_{r:n-1}(x)$are density function of the r^{th} order statistics in sample of size n-1 from the p-outlier model and the (p-i) outlier model respectively.

3.3 : RECURRENCE RELATIONS FOR NONIDENTICAL EXPONENTIAL RANDOM VARIABLE

Let us suppose X_1, X_2, ..., X_n are independent random variables and X_i has the exponential distribution with parameters $\lambda_i > 0$ i.e., X_i has the density $f_i(x) = \lambda_i e^{-\lambda_i x}$, $x > 0$, $i=1,2,...,n$ and the distribution function

$$F_i(x) = 1 - e^{-\lambda_i x}, \qquad x > 0, \quad i = 1, 2, ..., n.$$

$Y_1 \leq Y_2 \leq \ldots \leq Y_n$ denot e the corresponding order statistics. We first derive the m.g.f of Y_1, Y_2, ..., Y_n and then obtain a formula for the m.g.f of Y_r, $1 \leq r \leq n$, Which is best suited to derive the moment of Y_r. We also obtain the m.g.f of range $Y_n - Y_1$.

THEOREM 3.3.1 : (Bapat and Beg, 1989b)

Let $X_i \sim$ exponential (λ_i), $i = 1, 2, ..., n$ be independent then the m.g.f of Y_1, Y_2, ..., Y_n exists in a sufficiently small neighbourhood of the origin and is given by

$$\phi(t_1, t, \ldots, t_n) = (\prod_i \lambda_i)$$

$$\sum_{\sigma \varepsilon S_n} \frac{1}{(\lambda_{\sigma(n)} - t_n)(\lambda_{\sigma(n)} + \lambda_{\sigma(n-1)} - t_n - t_{n-1}) \ldots (\Sigma \lambda_i - \Sigma t_i)} \quad \ldots (3.3.1)$$

PROOF: The Joint density function of $Y_1, Y_2, \ldots, Y_n$ is given by

$$f(y_1, y_2, \ldots, y_n) = \sum_{\sigma \varepsilon S_n} \prod_i \left\{ \lambda_{\sigma(i)} e^{-\lambda_{\sigma(i)} y_i} \right\}, \quad 0 < y_1 < \ldots < y_n$$

Hence

$$\phi(t_1, t, \ldots, t_n) = E(e^{\Sigma t_i y_i}) = (\prod \lambda_i) \int_{y_1 < \ldots < y_n} \ldots \int e^{t_i y_i} \sum_{\sigma \varepsilon n} e^{-\Sigma \lambda_{\sigma(i)} y_i} \prod dy_i$$

$$= (\prod \lambda_i) \sum_{\sigma \varepsilon S_n} \int_0^{\infty} \ldots \int_{y_{n-2}}^{\infty} \int_{y_{n-1}}^{\infty} e^{-\Sigma(\lambda_{\sigma(i)} - t_i) y_i} \, d_{y_n} \ldots d_{y_1}$$

We get the result after routine integeration.

It is possible to obtain the m.g.f. of y_r by setting $t_i = 0$, $i \neq r$ in (3.3.1). Now we review another formula for the m.g.f. of y_r which can readily be used to calculate moments. Let us take $N = \{1, \ldots, n\}$. If $S \subset N$ then S' denote complement of S in N which $|S|$ denote the cardinality of S. If $S \subset N$, define

$$\lambda(S) = \sum_{i \in s} \lambda_i$$

THEOREM 3.3.2: (Bapat and Beg, 1989b)

Let $X_i \sim$ exponential (λ_i), $i = 1, \ldots, n$ be independent and let r be fixed, $1 \le r \le n$. Then the m.g.f. of y_r is given, for sufficiently small t, by

$$\phi(t) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{\lambda(s)}{\lambda(s)-t} \quad \ldots(3.3.2)$$

PROOF: The result can be proved by induction method. The result is trivial if n=1. Suppose the result is true for n-1. If r=1, then since Y_1 is exponential $(\sum \lambda_i)$, (3.5.2) clearly holds so suppose *r>1. Let* S_n^j *denote the set of permutations of the elements* of $N^j = \left\{ 1, \ldots, j-1, j+1, \ldots, n \right\}$.

The m.g.f. of Y_r is obtained, by setting $t_i=0$ for all $i \ne r$ in (3.5.1), as

$$\phi(t) = (\prod \lambda_i)$$

$$\sum_{\sigma \in S_n} \frac{1}{\lambda_{\sigma(n)} (\lambda_{\sigma(n)} + \lambda_{\sigma(n-1)}) \ldots (\lambda_{\sigma(n)} + \ldots + \lambda_{\sigma(r)} - t)\ (\sum \lambda_i - t)}$$

By induction hypothesis, we can write

$$\phi(t) = \frac{(\prod \lambda_i)}{\sum \lambda_i - t} \sum_{j=1, i \ne j}^{n} \frac{1}{(\prod \lambda_i)} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r}$$

$$\sum_{S \subset N^j, |S|=k} \frac{\lambda(s)}{\lambda(s)-t}$$

$$= \sum_{j=1}^{n} \lambda_j \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{S \subset N^j, |S|=k} \frac{\lambda(s)}{\lambda(s')}$$

$$\left\{\frac{1}{\lambda(s)-t} - \frac{1}{\sum \lambda_i - t}\right\} \qquad ...(3.3.3)$$

Consider $S \subset N$. If $|S| = k<n$, the coefficient of $\frac{\lambda(s)}{\lambda(s)-t}$ in (3.3.3) as well as in (3.3.2) is seen to be

$$(-1)^{k-n+r-1} \binom{k-1}{n-r}$$

Now we show that the coefficient of $(\sum \lambda_i - t)^{-1}$ is also identical in (3.3.3) and (3.3.2). The coefficient of $(\sum \lambda_i - t)^{-1}$ in (3.3.3) is

$$\sum_{k=n-r+1}^{n-1} (-1)^{k-n+r} \binom{k-1}{n-r} \sum_{|S|=k} \lambda(s)$$

$$= (\sum \lambda_i) \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r} \binom{k-1}{n-r} \binom{n-1}{k-1}$$

$$= \frac{(\sum \lambda_i)}{(n-r)!} \sum_{z=0}^{r-2} (-1)^{z+1} \frac{(n-1)!}{z!\,(r-z-1)!}$$

$$= (\sum \lambda_i)\,(-1)^{r-1} \binom{n-1}{n-r} \qquad ...(3.3.4)$$

Hence the last step follows from application of binomial theorem.

The coefficient of $(\sum \lambda_i - t)$ in (3.3.2) is also given by (3.3.4) there by proof is complete.

From (3.3.2), we obtain by differentiation,

$$E(Y_r) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{1}{\lambda(S)} \qquad \ldots(3.3.5)$$

$$E(Y_r^2) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{2}{[\lambda(S)]^2} \qquad \ldots(3.3.6)$$

From (3.3.5) and (3.3.6), we can get an expression for the variance of Y_r. In special case when $\lambda_1, \ldots\ldots, \lambda_{n-1}$ are equal, a different formula for the variance of Y_n has been obtained by Groos, Hunt and Odeh(1986).

If $X_i \sim$ exponential (1), i = 1, 2, ..., n are independent then it is well known (David 1981) that

$$E(Y_r) = \sum_{k=n-r+1}^{n} \frac{1}{k} \qquad \ldots(3.3.7)$$

We noticed that if $\lambda_i = 1$, i = 1, ..., n then $\lambda(S) = |S|$ for any $S \subset N$ and since there are $\binom{n}{k}$ subset of N of cardinality k,

$$\sum_{|S|=k} \frac{1}{\lambda(S)} = \binom{n}{k} \frac{1}{k}$$

Substituting above expression in (3.3.5) we get another expression of $E(y_r)$ and equate the following binomial identity.

$$\sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \binom{n}{k} \frac{1}{k} = \sum_{k=n-r+1} \frac{1}{k} \qquad \ldots(3.3.8)$$

For r = n of the identity of (3.3.8) has been mentioned by Feller(1968). But they hav e not been able to locate the general case.

THEOREM 3.3.3: (Bapat and Beg, 1989b)

Let $X_i \sim$ exponential (λ_i), $i = 1, 2, \ldots, n$ be independent and let r be fixed, $1 \le r \le n$. Then the m.g.f. of $Y_r - Y_1$ is given by

$$\phi(t) = \frac{1}{\Sigma \lambda_i} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{\lambda(S)\lambda(S')}{\lambda(S)-t} \quad \ldots(3.3.9)$$

PROOF: The m.g.f. of $Y_r - Y_1$ is obtained by setting $t_r = 1$, $t = -1$, $t_i = 0$, $i \neq 1, \ldots, r$ in (3.3.1)

$$\phi(t) = (\prod \lambda_i)$$

$$\sum_{\sigma \in S_n} \frac{1}{\lambda_{\sigma(n)} (\lambda_{\sigma(n)} + \lambda_{\sigma(n-1)}) \ldots (\lambda_{\sigma(n)} + \ldots + \lambda_{\sigma(r)} - t) \ldots (\Sigma \lambda_i)}$$

$$= \frac{\prod \lambda_i}{\Sigma \lambda_i} \sum_{j=1}^{n} \frac{1}{\prod_{i \neq j} \lambda_i} h_j(t) ,$$

By (3.3.1), $h_j(t)$ is the m.g.f. of the $(r-1)^{th}$ order statistics for the random variables $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$

By Theorem (3.3.2)

$$\phi(t) = \frac{\prod \lambda_i}{\Sigma \lambda_i} \sum_{j=1, i \neq j}^{n} \frac{1}{(\prod \lambda_i)} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{S \subset N^j, |S|=k} \frac{\lambda(s)}{\lambda(s)-t}$$

which complete the proof.

Put $r = n$ in Theorem(3.3.3) we obtain m.g.f. of the range $Y_n - Y_1$

from (3.3.9) as

$$\phi(t) = \frac{1}{\Sigma \lambda_i} \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{|S|=k} \frac{\lambda(S)\lambda(S')}{\lambda(S)-t} \qquad \ldots(3.3.10)$$

The raw moment of the range obtain from differentiation Of (3.3.10). Thus we have,

$$E\,(Y_n - Y_1) = \frac{1}{\Sigma \lambda_i} \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{|S|=k} \frac{\lambda(S')}{\lambda(S)}$$

Again we consider that the random variables X_i,s are independent having exponential distribution with density functions

$$f_i(x) = \frac{1}{\theta_i} e^{-x/\theta_i} \quad , \quad x \geq 0 \quad , \quad \theta_i > 0 \;, \qquad \ldots(3.3.11)$$

and d.f.

$$F_i(x) = 1 - e^{-x/\theta} \quad , \quad x \geq 0 \quad , \quad \theta_i > 0 \;, \qquad \ldots(3.3.12)$$

For $i = 1, 2, \ldots, n$, from (3.3.11) and (3.3.12) the distributions satisfy the differential equations.

$$f_i(x) = \frac{1}{\theta_i} \left[1 - F_i(x) \right] , \; x \geq 0 \;, \; \theta_i > 0 \;, \; i = 1, 2, \ldots, n \qquad \ldots(3.3.13)$$

Let us denote the single moments $E\,(x^k_{r:n})$ by $\mu^{(k)}_{r:n}$, $1 \leq r \leq n$ and $k = 1, 2, \ldots$ and product moments $E\,(x_{r:n}\,,\,x_{s:n})$ by $\mu_{r,s:n}$ for $1 \leq r < s \leq n$. we also use $\mu^{[i](k)}_{r,:n-1}$ and $\mu^{[i]}_{r,s:n-1}$ to denote the single and product moments of order statistics arising from n-1 variables obtained from deleting X_i from the original n variables $X_1, X_2, \ldots, X_n$. With the use of differential equation, we established several recurrence relations for single and product moments of order statistics and results for multiple outliers model are deduced as special case.

RELATIONS FOR SINGLE MOMENTS

THEOREM 3.3.4: (Balakrishnan ,1994)

For $n = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{\left(\sum_{i=1}^{n} (1/\theta_i)\right)} \mu_{1:n}^{(k)} \qquad \ldots(3.3.14)$$

THEOREM 3.3.5: (Balakrishnan ,1994)

For $2 \le r \le n$ and $k = 0, 1, 2, \ldots$

$$\mu_{r:n}^{(k+1)} = \frac{k+1}{\left(\sum_{i=1}^{n} (1/\theta_i)\right)} \left\{ (k+1)\, \mu_{r:n}^{(k)} + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1:n-1}^{[i](k+1)} \right\} \qquad \ldots(3.3.15)$$

RELATIONS FOR PRODUCT MOMENTS

THEOREM 3.3.6: (Balakrishnan ,1994)

For $n = 2, 3,$

$$\mu_{1,2:n} = \frac{1}{\left(\sum_{i=1}^{n} (1/\theta_i)\right)} \left\{ \mu_{1:n} + \mu_{2:n} \right\} \qquad \ldots(3.3.16)$$

THEOREM 3.3.7: (Balakrishnan ,1994)

For $2 \le r \le n-1,$

$$\mu_{r,r+1:n} = \frac{1}{\left(\sum_{i=1}^{n} (1/\theta_i)\right)} \left\{ \left(\mu_{r:n} + \mu_{r+1:n}\right) + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1,r:n-1}^{[i]} \right\} \qquad \ldots(3.3.17)$$

THEOREM 3.3.8: (Balakrishnan ,1994)

For $3 \le s \le n$,

$$\mu_{1,s:n} = \frac{1}{\left(\sum_{i=1}^{n} (1/\theta_i) \right)} \left\{ \mu_{1:n} + \mu_{s:n} \right\} \qquad \ldots(3.3.18)$$

THEOREM 3.3.9: For $2 \le r < s \le n$ & $s-r \ge 2$,

$$\mu_{r,s:n} = \frac{1}{\left(\sum_{i=1}^{n} (1/\theta_i) \right)} \left\{ \left(\mu_{r:n} + \mu_{s:n}\right) + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1,s-1:n-1}^{[i]} \right\} \qquad \ldots(3.3.19)$$

RELATIONS FOR p -OUTLIERS MODEL

Here we assume that X_1, X_2, ..., X_{n-p} are independent Exp(θ) random variables while X_{n-p+1}, ..., X_n are independent Exp(T) random variables and in independent of X_1, X_2, ..., X_{n-p} random variables. Here single moments denoted by $\mu_{r:n}^{(k)}$ [p] and the product moment denoted by $\mu_{r,s:n}$[p]. Similarly let us denote single and product moments by $\mu_{r:n-1}^{(k)}$[p-1]and $\mu_{r,s:n-1}$[p-1] respectively, when sample of size n-1 consists of p-1 outliers.

THEOREM 3.3.10: (Balakrishnan ,1994)

(a) For $n \ge 1$ and $k = 0, 1, 2,$

$$\mu_{1:n}^{(k+1)}[p] = \frac{k+1}{\left(\frac{n-p}{\theta} + \frac{p}{T} \right)} \mu_{1:n}^{(k)} [p] \qquad \ldots(3.3.20)$$

(b) For $2 \le r \le n$ and $k = 0, 1, 2,$

$$\mu_{r:n}^{(k+1)}[p] = \frac{1}{\left(\frac{n-p}{\theta} + \frac{p}{T} \right)}$$

$$\left\{ (k+1)\, \mu_{r:n}^{(k)} [p] \;+\; \frac{n-p}{\theta}\, \mu_{r-1:n-1}^{(k+1)}[p] \;+\; \frac{p}{T}\, \mu_{r-1:n-1}^{(k+1)}[p-1] \right\}$$

...(3.3.21)

(c) For $n \geq 2$,

$$\mu_{1,2:n}[p] = \frac{1}{\left[\dfrac{n-p}{\theta} + \dfrac{p}{T} \right]} \left\{ \mu_{1:n}[p] + \mu_{2:n}[p] \right\}$$

...(3.3.22)

(d) For $2 \leq r \leq n-1$,

$$\mu_{r,r+1:n}[p] = \frac{1}{\left[\dfrac{n-p}{\theta} + \dfrac{p}{T} \right]}$$

$$\left\{ \mu_{r:n}[p] + \mu_{r+1:n}[p] + \frac{n-p}{\theta}\, \mu_{r-1,r:n-1}[p] + \frac{p}{T}\, \mu_{r-1,r:n-1}[p-1] \right\}$$

...(3.3.23)

(e) for $3 \leq s \leq n$,

$$\mu_{1,s:n}[p] = \frac{1}{\left[\dfrac{n-p}{\theta} + \dfrac{p}{T} \right]} \left\{ \mu_{1:n}[p] + \mu_{s:n}[p] \right\}$$

...(3.3.24)

(f) For $2 \leq r < s \leq n$ and $s-r \geq 2$,

$$\mu_{1,s:n}[p] = \frac{1}{\left[\dfrac{n-p}{\theta} + \dfrac{p}{T} \right]}$$

$$\left\{ \mu_{r:n}[p] + \mu_{s:n}[p] + \frac{n-p}{\theta}\, \mu_{r-1,s-1:n-1}[p] + \frac{p}{T}\, \mu_{r-1,s-1:n-1}[p-1] \right\}$$

...(3.3.25)

REMARK 3.3.1: The recurrence relation in (3.3.20) to (3.3.25) will enable one to compute all single amd product moments all order statistics from p-outlier model in the simple recurrence manner . For instant

$$\mu_{r:n}[0] = \theta \sum_{i=1}^{r} \frac{1}{n-i+1} ,$$

$$\mu_{r:n}^{(2)}[0] = \theta^2 \left[\sum_{i=1}^{r} \frac{1}{(n-i+1)^2} + \left(\sum_{i=1}^{r} \frac{1}{n-i+1} \right)^2 \right] ,$$

and

$$\mu_{r,s:n}^{(2)}[0] = \theta^2 \left[\sum_{i=1}^{r} \frac{1}{(n-i+1)^2} + \left(\sum_{i=1}^{r} \frac{1}{n-i+1} \right) \left(\sum_{j=1}^{r} \frac{1}{n-j+1} \right) \right]$$

Similarly recurrence relations for the first two single moments and product moments of all order statistics from sample containing two outlers and ~~so on~~, can be obtained.

3.4 RECUERRENCE RELATIONS FOR PRODUCTS OF MOMENTS AND PRODUCT MOMENTS

THEOREM 3.4.1: (Balasubramanian and Beg, Pre-Printed, 1995)

for $1 \le r \le n_1$, $1 \le s \le n_2$ and $n = n_1 + n_2$,

$$\sum_{|S|=n_1} E\, g_1(x_{r:S})\, E\, g_2(x_{s:S'}) = \sum_{j=0}^{n_1-r} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{n-r-j-k-1}{n_1-r-j,\, n_2-s,\, s-k-1}$$

$$\sum_{|S|=n-k} E \left\{ g_1(x_{r:S})\, g_2(x_{r+j-1:S}) \right\} \sum_{j=0}^{n_2-s} \sum_{k=0}^{r-1} (-1)^{r-k-1}$$

$$\binom{n-s-j-k-1}{n_2-s-j,\, n_1-r,\, r-k-1} \sum_{|S|=n-k} E \left\{ g_2(x_{s:S})\, g_1(x_{s+j-1:S}) \right\} \qquad \ldots(3.4.1)$$

PROOF: Using (1.9.3),

$$\sum_{|S|=n_1} E\, g_1(x_{r:S})\, E\, g_2(x_{s:S'}) = \sum_{|S|=n_1} \frac{1}{(r-1)!(n_1-r)} \int_{-\infty}^{\infty} g_1(x)$$

$$\mathrm{Per}\,[\underset{r-1}{\underline{F(x)}}, \underset{n_1-r}{\underline{1-F(x)}}, \underset{1}{\underline{f(x)}}]\,[S|.)\,dx \frac{1}{(s-1)!(n_2-s)} \int_{-\infty}^{\infty} g_2(y)$$

$$\mathrm{Per}\,[\underset{s-1}{\underline{F(y)}}, \underset{n_2-s}{\underline{1-F(y)}}, \underset{1}{\underline{f(y)}}]\,[S|.)\,dy$$

$$= \frac{1}{(r-1)!(n_1-r)(s-1)!(n_2-s)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)\, g_2(y)$$

$$\mathrm{Per}\,[\underset{r-1}{\underline{F(x)}}, \underset{n_1-r}{\underline{1-F(x)}}, \underset{1}{\underline{f(x)}}, \underset{s-1}{\underline{F(y)}}, \underset{n_2-s}{\underline{1-F(y)}}, \underset{1}{\underline{f(y)}}]\,dx\,dy$$

$$= \frac{1}{(r-1)!(n_1-r)(s-1)!(n_2-s)} \left\{ \int\int_{x<y} + \int\int_{y<x} \right\}$$

$$= \frac{1}{(r-1)!(n_1-r)(s-1)!(n_2-s)} \left\{ I_1 + I_2 \right\}, \text{ say} \qquad \ldots(3.4.2)$$

we evaluate I_1 and I_2,

$$I_1 = \int\int_{x<y} g_1(x)\, g_2(y)$$

$$\mathrm{Per}\,[\underset{r-1}{\underline{F(x)}}, \underset{n_1-r}{\underline{1-F(x)}}, \underset{s-1}{\underline{F(y)}}, \underset{n-s}{\underline{1-F(y)}}, \underset{1}{\underline{f(x)}}, \underset{1}{\underline{f(y)}}]\,dx\,dy$$

$$I_1 = \int\int_{x<y} g_1(x)\, g_2(y)\ \mathrm{Per}\,[\underset{r-1}{\underline{F(x)}}, \underset{n_1-r}{\underline{(F(y)-F(x)+(1-F(y))}},$$

$$\underset{s-1}{\underline{(\ 1-(1-F(y))\ )}},\ \underset{n_2-s}{\underline{1-F(y)}},\ \underset{1}{\underline{f(x)}},\ \underset{1}{\underline{f(y)}}\ ]\ dx\ dy$$

$$= \sum_{j=0}^{n_1-r} \binom{n_1-r}{j} \int\int_{x<y} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\ [\underset{r-1}{\underline{F(x)}},\ \underset{j}{\underline{F(y)-F(x)}},\ \underset{k}{\underline{1}},\ \underset{n_1-r-j}{\underline{1-F(y)}},\ \underset{s-1}{\underline{1-(1-F(y)}}\ \underset{1}{\underline{f(x)}},\ \underset{1}{\underline{f(y)}}\ ]\ dx\ dy$$

$$= \sum_{j=0}^{n_1-r} \binom{n_1-r}{j} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} \int\int_{x<y} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\ [\ \underset{r-1}{\underline{F(x)}},\ \underset{j}{\underline{F(y)-F(x)}},\ \underset{k}{\underline{1}},\ \underset{n-r-j-k-1}{\underline{1-F(y)}};\ \underset{1}{\underline{f(x)}},\ \underset{1}{\underline{f(y)}}\ ]\ dx\ dy$$

$$= \sum_{j=0}^{n_1-r} \binom{n_1-r}{j} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} \sum_{|S|=n-k} k! \int\int_{x<y} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\ [\ \underset{r-1}{\underline{F(x)}},\ \underset{j}{\underline{F(y)-F(x)}},\ \underset{n-r-j-k-1}{\underline{1-F(y)}},\ \underset{1}{\underline{f(x)}},\ \underset{1}{\underline{f(y)}}\ ]\ [S|.)\ dx\ dy$$

$$= \sum_{j=0}^{n_1-r} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{n_1-r}{j} \binom{s-1}{k} \sum_{|S|=n-k} k!$$

$$E\left\{g_1(x_{r:S})\ g_2(x_{r+j+1:S})\right\}\ (r-1)!\ j!\ (n-r-j-k-1)!$$

Similarly, using (1.9.3)

$$I_2 = \int\int_{y<x} g_1(x)\ g_2(y)$$

$$\text{Per}\left[\frac{F(y)}{s-1}, \frac{1-F(y)}{n-s}, \frac{f(y)}{1} \frac{F(x)}{r-1}, \frac{1-F(x)}{n_1-r}, \frac{f(x)}{1} \right] dx\, dy$$

$$= \sum_{j=0}^{n_2-r} \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{n_2-s}{j} \binom{r-1}{k} \sum_{|S|=n-k} k!$$

$$E\left\{ g_2(x_{s:S})\, g_1(x_{s+j+1:S}) \right\} \quad (s-1)!\; j!\; (n-s-j-k-1)!$$

subtituting I_1 and I_2 in (3.4.1) and simplifying, we get the result of Theorem (3.~~4.~~1.).

COROLLARY 3.4.1: If $g_2(x) = 1$, Theorem 3.4.1. yields

$$\sum_{|S|=n_1} E\, g_1(x_{r:S}) = \sum_{j=0}^{n_1-r} \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{n-r-j-k-1}{n_1-r-j,\, n_2-s,\, s-k-1}$$

$$\sum_{|S|=n-k} E\, g_1(x_{r:S}) + \sum_{j=0}^{n_2-s} \sum_{k=0}^{r-1} (-1)^{r-k-1}$$

$$\binom{n-s-j-k-1}{n_2-s-j,\, n_1-r,\, r-k-1} \sum_{|S|=n-k} E\left\{ g_1(x_{s+j+1:S}) \right\} \qquad \ldots(3.4.3)$$

Here it is to be noted that if we choose $g_1(x)$ and $g_2(x)$ suitably then Theorem(3.4.1) yields identities involving moment generating functions, characteristic functions and distribution functions.

THEOREM 3.4.2: (Beg, 1991)

For $1 \le r < s \le n$,

$$E\left\{ g_1(x_{r:n})\ g_2(x_{s:n}) \right\} + \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{j+k}{k} \sum_{|S|=n-j-k}$$

$$E\left\{ g_2(x_{n-s-j+1:S})\ g_1(x_{n-r-j+1:S}) \right\} = \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-i-1}{r-1}$$

$$\sum_{|S|=S-i} E\left\{ g_1(x_{s-i:S}) \right\} E\left\{ g_2(x_{1:S'}) \right\} \qquad \ldots(3.4.4)$$

PROOF: Consider $I = (\ (r-1)!\ (s-r-1)1\ (n-s)!\ )^{-1} J$, where

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\left[\frac{F(x)}{r-1}, \frac{F(y)-F(x)}{s-r-1}, \frac{1-F(y)}{n-s}, \frac{f(x)}{1}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \sum_{t=0}^{s-r-1} (-1)^{s-r-t-1} \binom{s-r-1}{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\left[\frac{F(x)}{s-t-2}, \frac{f(x)}{1}, \frac{F(y)}{t}, \frac{1-F(y)}{n-s}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \sum_{t=0}^{s-r-1} (-1)^{s-r-t-1} \binom{s-r-1}{t} \sum_{|S|=s-t-1} \int_{-\infty}^{\infty} g_1(x)\ \mathrm{Per}\left[\frac{F(x)}{s-t-2}, \frac{f(x)}{1} \right]$$

$$[S|.)dx \int_{-\infty}^{\infty} g_2(y)\ \mathrm{Per}\left[\frac{F(y)}{t}, \frac{1-F(y)}{n-s}, \frac{f(y)}{1} \right] [S'|.)\ dy$$

$$= \sum_{t=0}^{s-r-1} (-1)^{s-r-t-1} \binom{s-r-1}{t} \sum_{|S|=s-t-1} (s-t-2)!\ t!\ (n-s)!$$

$$E\left\{ g_1(x_{s-t-1:S}) \right\} E\left\{ g_2(x_{t+1:S'}) \right\}$$

Using (1.9.3) and writing i = t+1, we get

$$J = \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-r-1}{i-1} \sum_{|S|=s-i} (s-i-1)!\ (i-1)!\ (n-s)!$$

$$E\left\{ g_1(x_{s-i:S}) \right\} E\left\{ g_2(x_{i:S'}) \right\}$$

and

$$I = \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-r-1}{i-1} \sum_{|S|=s-i} E\left\{ g_1(x_{s-i:S}) \right\} E\left\{ g_2(x_{i:S'}) \right\}$$

which is the RHS (3.4.4). Further

$$I = (\ (r-1)!\ (s-r-1)1\ (n-s)!\ )^{-1} \Bigg[\int\int_{x<y} g_1(x)\ g_2(y)$$

$$\text{Per}\ [\ \underset{r-1}{\underline{F(x)}}\ ,\ \underset{s-r-1}{\underline{F(y) - F(x)}}\ ,\ \underset{n-s}{\underline{1-F(y)}}\ ,\ \underset{1}{\underline{f(x)}}\ ,\ \underset{1}{\underline{f(y)}}\ ]\ dx\ dy$$

$$+ \int\int_{y<x} g_1(x)\ g_2(y)$$

$$\text{Per}\ [\ \underset{r-1}{\underline{F(x)}}\ ,\ \underset{s-r-1}{\underline{F(y) - F(x)}}\ ,\ \underset{n-s}{\underline{1-F(y)}}\ ,\ \underset{1}{\underline{f(x)}}\ ,\ \underset{1}{\underline{f(y)}}\ ]\ dx\ dy \Bigg]$$

$$= E\left\{ g_1(x_{r:n})\ g_2(x_{s:n}) \right\} + (\ (r-1)!\ (s-r-1)1\ (n-s)!\ )^{-1} J_1$$

where

$$J_1 = \iint_{y<x} g_1(x)\; g_2(y)$$

$$\mathrm{Per}\left[\frac{F(x)}{r-1},\ \frac{F(y)-F(x)}{s-r-1},\ \frac{1-F(y)}{n-s},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] dx\, dy$$

$$= (-1)^{s-r-1} \iint_{y<x} g_1(x)\; g_2(y)$$

$$\mathrm{Per}\left[\frac{1-(1-F(x))}{r-1},\ \frac{F(x)-F(y)}{s-r-1},\ \frac{1-F(y)}{n-s},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] dx\, dy$$

$$= \sum_{k=0}^{r-1} (-1)^{s-k-2} \binom{r-1}{k} \iint_{y<x} g_1(x)\; g_2(y)$$

$$\mathrm{Per}\left[\frac{1}{k},\ \frac{1-F(x)}{r-k-1},\ \frac{F(x)-F(y)}{s-r-1},\ \frac{1-F(y)}{n-s},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] dx\, dy$$

$$= \sum_{k=0}^{r-1} (-1)^{s-k-2} \binom{r-1}{k} \sum_{j=0}^{n-s} (-1)^{n-s-j} \binom{n-s}{j} \iint_{y<x} g_1(x)\; g_2(y)$$

$$\mathrm{Per}\left[\frac{1}{j+k},\ \frac{F(y)}{n-s-j},\ \frac{F(x)-F(y)}{s-r-1},\ \frac{F(x)}{r-k-1},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] dx\, dy$$

$$= \sum_{k=0}^{r-1} \sum_{j=0}^{n-s} (-1)^{n-j-k} \binom{r-1}{k} \binom{n-s}{j} \sum_{|S|=n-j-k} (j+k)! \iint_{y<x} g_1(x)\; g_2(y)$$

$$\mathrm{Per}\left[\frac{F(y)}{n-s-j},\ \frac{F(x)-F(y)}{s-r-1},\ \frac{F(x)}{r-k-1},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] [S|\,.)\ dx\, dy$$

$$= \sum_{j=0}^{n-s} \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{r-1}{k} \binom{n-s}{j} (j+k)! \sum_{|S|=n-j-k}$$

$$E\left\{ g_2(x_{n-s-j+1:S})\; g_1(x_{n-r-j+1:S}) \right\} (n-s-j)!\ (s-r-1)!\ (r-k-1)!,$$

Using (1.9.3) and with the simplification we get I equal to

LHS Of (3.4.4)

COROLLARY 3.4.2: If $g_2(x) = 1$, Theorem(3.4.2) yields

$$E\left\{g_1(x_{r:n})\right\} + \sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{j+k}{k}\sum_{|S|=n-j-k} E\left\{g_1(x_{n-r-j+1:S})\right\}$$

$$= \sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\sum_{|S|=s-i} E\left\{g_1(x_{s-i:S})\right\} \qquad \ldots(3.4.5)$$

which is a recurrence relation involving single moments of functions of order statistics.

COROLLARY 3.4.3: For the case of a sample of n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ having pdf f(x) and cdf F(x) Theorem(3.4.2) reduces

$$E\left\{ g_1(x_{r:n})\, g_2(x_{s:n}) \right\} + \sum_{j=0}^{n-s}\sum_{k=0}^{r-1}(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{k}$$

$$E\left\{ g_2(x_{n-s-j+1:n-j-k})\, g_1(x_{n-r-j+1:n-j-k}) \right\}$$

$$= \sum_{i=1}^{s-r}(-1)^{s-r-i}\binom{s-i-1}{r-1}\binom{n}{s-i}$$

$$E\left\{ g_1(x_{s-i:s-i}) \right\} E\left\{ g_2(x_{i:n-s+i}) \right\} \qquad \ldots(3.4.6)$$

COROLLARY 3.4.4: For the p-Outlier model, that is, $F_1 = F_2 = \ldots. = F_{n-p} = F$ and $F_{n-p+1} = \ldots = F_n = G$ (Outlier Distribution) Theorem(3.4.2) yields

$$E\left\{g_1(x_{r:n})\ g_2(x_{s:n})\right\} + \sum_{j=0}^{n-s}\ \sum_{k=0}^{r-1} (-1)^{n-j-k} \binom{j+k}{k} \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{n-j-k-a}$$

$$E\left\{ g_2(x_{n-s-j+1:n-j-k,a})\ g_1(x_{n-r-j+1:n-j-k,a}) \right\}$$

$$= \sum_{i=1}^{s-r} (-1)^{s-r-i} \binom{s-i-1}{r-1} \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{n-j-k-a}$$

$$E\left\{ g_1(x_{s-i:s-i,a}) \right\} E\left\{ g_2(x_{i:n-s+i,p-a}) \right\} \qquad \ldots(3.4.7)$$

where $X_{r:n,a}$ denote the r^{th} order statistics from a sample of size n of which 'a' are outliers.

3.5 IDETITIES FOR PRODUCT MOMENT

THEOREM 3.5.1: (Beg, 1991)

For $1 \le i \le n-2$

$$\sum_{|S|=n-i} i!\ E\left\{ g_1(x_{n-i-1:S})\ g_2(x_{n-i:S}) \right\} = \frac{1}{(n-i-2)!} \sum_{j=0}^{i}\ \sum_{k=0}^{j} \binom{i}{j}$$

$$j!\ (n-j-2)!\ E\left\{ g_1(x_{n-j-1:n})\ g_2(x_{n-k:n}) \right\} \qquad \ldots(3.5.1)$$

PROOF:

$$\sum_{|S|=n-i} i!\ E\left\{ g_1(x_{n-i-1:S})\ g_2(x_{n-i:S}) \right\} = \sum_{|S|=n-i} \frac{i!}{(n-i-2)!}$$

$$\int\!\!\int_{x<y} g_1(x)\ g_2(y)\ \text{Per}\left[\underset{n-i-2}{\underline{F(x)}}\ ,\ \underset{1}{\underline{f(x)}}\ ,\ \underset{1}{\underline{f(y)}}\ \right] [S|.)\ dx\ dy$$

$$= \frac{1}{(n-i-2)!} \int\int_{x<y} g_1(x)\, g_2(y)\ \mathrm{Per}\left[\frac{1}{i}, \frac{F(x)}{n-i-2}, \frac{f(x)}{1}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \frac{1}{(n-i-2)!} \int\int_{x<y} g_1(x)\, g_2(y)$$

$$\mathrm{Per}\left[\frac{(F(x) + (F(y)-F(x)) + (1-F(y)),}{i} \frac{F(x)}{n-i-2}, \frac{f(x)}{1}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \frac{\sum_{j=0}^{i} \binom{i}{j}}{(n-i-2)!} \int\int_{x<y} g_1(x)\, g_2(y)$$

$$\mathrm{Per}\left[\frac{(F(y)-F(x)) + (1-F(y)),}{j} \frac{F(x)}{n-j-2}, \frac{f(x)}{1}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \frac{\sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k}}{(n-i-2)!} \int\int_{x<y} g_1(x)\, g_2(y)$$

$$\mathrm{Per}\left[\frac{F(x)}{n-j-2}, \frac{(F(y)-F(x)),}{j-k} \frac{(1-F(y)),}{k} \frac{f(x)}{1}, \frac{f(y)}{1} \right] dx\, dy$$

$$= \frac{\sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j}\binom{j}{k}}{(n-i-2)!} E\left\{ g_1(x_{n-j-1:n})\, g_2(x_{n-k:n}) \right\} (n-j-2)!(n-k)!k!$$

$$= \sum_{j=0}^{i} \sum_{k=0}^{j} \frac{\binom{i}{j}\ j!\ (n-j-2)!}{(n-j-2)!} E\left\{ g_1(x_{n-j-1:n})\, g_2(x_{n-k:n}) \right\}$$

thus proof is completed.

We have following corollaries corresponding to the corollaries

(3.4.1) to (3.4.3).

COROLLARY 3.5.1: $\sum_{|S|=n-j} i!\, E\left\{ g_1(x_{n-i-1:S}) \right\}$

$$= \frac{\sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} j!\,(n-j-2)!}{(n-i-2)!} E\left\{ g_1(x_{n-j-1:n}) \right\} \qquad \ldots(3.5.2)$$

COROLLARY 3.5.2:

$$(n-i)\,(n-i-1) \sum_{j=0}^{i} \left[\frac{\binom{i}{j}}{\binom{n-2}{j}} \right] \sum_{k=0}^{j} E\left\{ g_1(x_{n-j-1:n})\, g_2(x_{n-k:n}) \right\} -$$

$$= n\,(n-1)\, E\left\{ g_1(x_{n-i-1:n-i})\, g_2(x_{n-i:n-j}) \right\} \qquad \ldots(3.5.3)$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r,s:n} = E\,(X_{r:n}\, X_{s:n})$, (3.7.3) reduces to theorem 3.1 of Joshi and Balakrishnan (1982).

COROLLARY 3.5.3:

$$i! \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{n-i-a} E\left\{ g_1(x_{n-i-1:n-i,a})\, g_2(x_{n-i:n-1,a}) \right\}$$

$$= \frac{\sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} j!\,(n-j-2)!}{(n-i-2)!} E\left\{ g_1(x_{n-j-1:n})\, g_2(x_{n-k:n}) \right\} \qquad \ldots(3.5.4)$$

THEOREM 3.5.2: (Beg, 1991)

For $1 \le r < n$

$$\sum_{r=1}^{n-1} E\left\{ g_1(X_{r:n})\ g_2(X_{r+1:n}) \right\} + \sum_{j=2}^{n} \sum_{|S|=j} E\left\{ g_2(X_{1:S})\ g_1(X_{j:S}) \right\}$$

$$= \sum_{j=1}^{n-1} \sum_{|S|=j} E\left\{ g_1(X_{j:S}) \right\} E\left\{ g_2(X_{1:S'}) \right\} \qquad \ldots(3.5.5)$$

PROOF : Consider the sum of integrals

$$I = \sum_{j=1}^{n-1} \frac{1}{(j-1)!\ (n-j-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x)\ g_2(x)$$

$$\operatorname{Per}\left[\underset{j-1}{\underline{F(x),}} \quad \underset{n-j-1}{\underline{1-F(y),}} \quad \underset{1}{\underline{f(x),}} \quad \underset{1}{\underline{f(y)}} \right] dx\ dy$$

$$= \sum_{j=1}^{n-1} \frac{1}{(j-1)!\ (n-j-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x)\ g_2(y)$$

$$\sum_{|S|=j} \operatorname{Per}\left[\underset{j-1}{\underline{F(x)}} \quad \underset{1}{\underline{f(x)}} \right] [S|.)\ \operatorname{Per}\left[\underset{n-j-1}{\underline{1-F(y)}} \quad \underset{1}{\underline{f(y)}} \right] [S'|.)\ dx\ dy$$

$$= \sum_{j=1}^{n-1} \frac{1}{(j-1)!\ (n-j-1)!} \sum_{|S|=j} \int_{-\infty}^{\infty} g_1(x)\ \operatorname{Per}\left[\underset{j-1}{\underline{F(x)}} \quad \underset{1}{\underline{f(x)}} \right] [S|.)\ dx$$

$$\int_{-\infty}^{\infty} g_2(y)\ \operatorname{Per}\left[\underset{n-j-1}{\underline{1-F(y)}} \quad \underset{1}{\underline{f(y)}} \right] [S'|.)\ dy$$

$$= \sum_{j=1}^{n-1} \sum_{|S|=j} E\left\{ g_1(x_{j:S}) \right\} E\left\{ g_2(x_{1:S'}) \right\}$$

which is the RHS of (3.5.5). Further we can write

$$I = \sum_{j=1}^{n-1} \frac{1}{(j-1)!\ (n-j-1)!} \left[\int\int_{x<y} g_1(x)\ g_2(y) \right.$$

$$\mathrm{Per}\ \left[\frac{F(x)}{j-1}\ ,\ \frac{1-F(y)}{n-j-1}\ ,\ \frac{f(x)}{1}\ ,\ \frac{f(y)}{1} \right] dx\ dy\ +\ \int\int_{y<x} g_1(x)\ g_2(y)$$

$$\left. \mathrm{Per}\ \left[\frac{F(x)}{j-1}\ ,\ \frac{1-F(y)}{n-j-1}\ ,\ \frac{f(x)}{1}\ ,\ \frac{f(y)}{1} \right] dx\ dy \right]$$

$$= \sum_{j=1}^{n-1} E\left\{ g_1(x_{j:n}) \right\}\ E\left\{ g_2(x_{j+1:n}) \right\}\ +\ J$$

where

$$J = \sum_{j=1}^{n-1} \frac{1}{(j-1)!\ (n-j-1)!} \int\int_{y<x} g_1(x)\ g_2(y)$$

$$\mathrm{Per}\ \left[\frac{F(x)}{j-1}\ ,\ \frac{1-F(y)}{n-j-1}\ ,\ \frac{f(x)}{1}\ ,\ \frac{f(y)}{1} \right] dx\ dy$$

$$= \frac{\quad}{(n-2)!} \int\int_{y<x} g_1(x)\ g_2(y) \sum_{j=1}^{n-1} \binom{n-2}{j-1}$$

$$\mathrm{Per}\ \left[\frac{F(x)}{j-1}\ ,\ \frac{1-F(y)}{n-j-1}\ ,\ \frac{f(x)}{1}\ ,\ \frac{f(y)}{1} \right] dx\ dy$$

$$= \frac{1}{(n-2)!} \int\int_{y<x} g_1(x)\ g_2(y)\ \mathrm{Per}\ \left[\frac{(F(x)+1-F(y))}{n-2},\ \frac{f(x)}{1},\ \frac{f(y)}{1} \right] dx\ dy$$

$$= \frac{1}{(n-2)!} \int\int_{y<x} g_1(x)\ g_2(y) \sum_{j=1}^{n-1} \binom{n-2}{j-1}$$

$$\text{Per}\left[\frac{1}{k},\ \frac{(F(x)-F(y))}{n-k-2},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right] dx\, dy$$

$$= \frac{1}{(n-2)!}\sum_{j=1}^{n-1}\binom{n-2}{k}\sum_{|S|=n-k} k!\int\int_{y<x} g_1(x)\ g_2(y)$$

$$\text{Per}\left[\frac{(F(x)-F(y))}{n-k-2},\ \frac{f(x)}{1},\ \frac{f(y)}{1}\right][S|.)\ dx\, dy$$

$$= \frac{1}{(n-2)!}\sum_{j=1}^{n-1}\binom{n-2}{k}\ k!\sum_{|S|=n-k} E\left\{g_2(x_{1:S})\ g_1(x_{n-k:S})\right\}\ (n-k-2)!$$

Puting $j = n-k$, we get

$$J = \sum_{j=2}^{n}\ \sum_{|S|=j} E\left\{g_2(x_{1:S})\right\}\ E\left\{g_1(x_{j:S})\right\}$$

Here, we see that I is also equal to LHS of (3.5.5) thereby complete the proof.

We have following corollaries corresponding to the corollaries (3.4.1) to (3.4.3).

COROLLARY 3.5.4:

$$\sum_{j=1}^{n-1} E\left\{g_1(x_{r:n})\ g_2(x_{r+1:n})\right\} + \sum_{j=2}^{n}\binom{n}{j} E\left\{g_2(x_{1:j})\ g_1(x_{j:j})\right\}$$

$$= \sum_{j=1}^{n-1}\binom{n}{j} E\left\{g_1(x_{j:j})\ g_2(x_{1:n-j})\right\} \qquad \ldots(3.5.6)$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r:n} = E\left\{x_{r:n}\right\}$ and $\mu_{r,s:n} = E\left\{x_{r:n}\ x_{s:n}\right\}$, then (3.5.6) reduce to Theorem 3.2 of

Balakrishnan(1982).

COROLLARY 3.5.5:

$$\sum_{j=1}^{n-1} E\left\{ g_1(x_{r:n}) \right\} \sum_{j=1}^{n-1} \sum_{|S|=j} E\left\{ g_1(x_{j:S}) \right\}$$

$$= \sum_{j=2}^{n} \sum_{|S|=j} E\left\{ g_1(x_{j:S}) \right\} \qquad \ldots(3.5.7)$$

COROLLARY 3.5.6:

$$\sum_{r=1}^{n-1} E\left\{ g_1(x_{r:n})\; g_2(x_{r+1:n}) \right\} + \sum_{j=2}^{n} \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{j-a}$$

$$E\left\{ g_2(x_{1:j,a})\; g_1(x_{j:j,a}) \right\} = \sum_{j=1}^{n-1} \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{j-a}$$

$$E\left\{ g_1(x_{j:j,a}) \right\} E\left\{ g_2(x_{1:j}) g_1(x_{j:j}) \right\} \qquad \ldots(3.5.8)$$

THEOREM 3.5.3: (Beg, 1991)

For $1 \le r < s \le n$ and $1 < k < n-2$

$$\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\left\{ g_1(x_{r:n})\; g_2(x_{s:n}) \right\} + \sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1} \binom{n-s}{n-k-r}$$

$$E\left\{ g_2(x_{i:n})\; g_1(x_{s:n}) \right\} = \sum_{|S|=n-k} E\left\{ g_1(x_{r:S}) \right\} E\left\{ g_2(x_{1:S'}) \right\} \qquad \ldots(3.5.9)$$

PROOF: Its proof is identical with Theorem(3.5.1) and Theorem(3.5.2).

COROLLARY 3.5.7:

$$\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\left\{ g_1(x_{r:n}) \right\} + \sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1} \binom{n-s}{n-k-r}$$

$$E\left\{ g_1(x_{s:n}) \right\} = \sum_{|S|=n-k} E\left\{ g_1(x_{r:S}) \right\} \qquad \ldots(3.5.10)$$

COROLLARY 3.5.8:

$$\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\left\{ g_1(x_{r:n})\, g_2(x_{s:n}) \right\} + \sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1} \binom{n-s}{n-k-r}$$

$$E\left\{ g_2(x_{i:n})\, g_1(x_{s:n}) \right\} = \binom{n}{n-k} E\left\{ g_1(x_{r:n}) \right\} E\left\{ g_2(x_{1:k}) \right\} \qquad \ldots(3.5.11)$$

Taking $g_1(x) = g_2(x) = x$ and writing $\mu_{r:n} = E\left\{x_{r:n}\right\}$ and $\mu_{r,s:n} = E\left\{x_{r:n}\, x_{s:n}\right\}$, then (3.5.6) reduce to Theorem 3.3 of Balakrishnan(1982).

COROLLARY 3.5.9:

$$\sum_{s=r+1}^{n-k+1} \binom{n-s}{k-1} E\left\{ g_1(x_{r:n})\, g_2(x_{s:n}) \right\} + \sum_{i=1}^{r} \sum_{s=r+1}^{r+k} \binom{s-i-1}{s-r-1} \binom{n-s}{n-k-r}$$

$$E\left\{ g_2(x_{i:n})\, g_1(x_{s:n}) \right\} = \sum_{a=0}^{p} \binom{p}{a} \binom{n-p}{n-k-r}$$

$$E\left\{ g_1(x_{r:n-k,a}) \right\} E\left\{ g_2(x_{1:k,p-a}) \right\} \qquad \ldots(3.5.12)$$

3.6 RECURRENCE RELATIONS FOR SINGLE MOMENTS IN RIGHT TRUNCATED EXPONENTIAL DISTRIBUTION

We consider the case when the variables X_i's are independent having right truncated exponential distribution with density functions

$$f_i(x) = \frac{1}{\theta_i\,(1 - e^{T/\theta_i})}\, e^{-x/\theta_i} \;, \qquad 0 \le x \le T \;,\; \theta_i > 0 \qquad \ldots(3.6.1)$$

$$F\,(x) = \frac{1 - e^{-x/\theta_i}}{1 - e^{T/\theta_i}} \;, \qquad 0 \le x \le T \;,\; \theta_i > 0 \qquad \ldots(3.6.2)$$

for $i = 1, 2, \ldots, n$. T denotes the point of truncation on the right of the standard exponential distribution from (3.6.1) and (3.6.2).

$$f_i(x) = \frac{1}{\theta_i}\left\{ 1 - F_i(x) \right\} + S_i \;, \qquad 0 \le x \le T \;,\; \theta_i > 0 \qquad \ldots(3.6.3)$$

for $i = 1, 2, \ldots, n.$ where

$$S_i = \frac{e^{-T/\theta}}{\theta_i\,(1 - e^{-T/\theta_i})} \qquad \ldots(3.6.4)$$

with the differential (3.6.3), we rewiew several recurrence relations for single moments established by Balakrishnan(1994).

THEOREM 3.6.1: (Balakrishnan, 1994)

For $k = 0, 1, 2, \ldots$,

$$\mu_{1:1}^{(k+1)} = \frac{1}{1/\theta_1}\left\{ (k+1)\,\mu_{1:1}^{k} - S_1\, T^{k+1} \right\} \qquad \ldots(3.6.5)$$

where S_1 is as define in (3.6.4).

PROOF: For $k = 0, 1, 2, \ldots,$ Let us consider

$$\mu_{1:1}^{(k)} = \int_0^T x^k \; f_1(x)\, dx = \frac{1}{\theta_1} \int_0^T x^k \left\{ 1 - F_1(x) \right\} dx$$

$$+ \quad S_1 \int_0^T x^k \, dx$$

using (3.6.3) and integerating by parts treating x^k for integeration , we obtain

$$\mu_{1:1}^{(k)} = \frac{1}{k+1} \left\{ \frac{1}{\theta_1} \; \mu_{1:1}^{(k+1)} + S_1 \, T^{k+1} \right\}$$

After rearranging we find the result. —

THEOREM 3.6.2: (Balakrishnan, 1994)

For $1 \le r \le n-1$ and $k = 0, 1, \ldots,$

$$\mu_{r:n}^{(k+1)} = \frac{1}{\left(\sum_{i=1}^{n} 1/\theta_i \right)} \left\{ (k+1)\, \mu_{r:n}^{(k)} + \sum_{i=1}^{n} \left(\frac{1}{\theta_i} + S_i \right) \mu_{r-1:n-1}^{[i](k+1)} - \sum_{i=1}^{n} S_i \, \mu_{r:n-1}^{[i](k+1)} \right\}, \qquad \ldots(3.6.6)$$

with the convention that $\mu_{0:n-1}^{[i](k+1)} \equiv 0.$

PROOF: From (1.9.1), let us consider for $1 \le r \le n-1$ and $k = 0, 1, 2, \ldots,$

$$(r-1)!(n-r)!\; \mu_{r:n}^{(k)} = \sum_p \int_0^T x^k \prod_{a=1}^{r-1} F_{i_a}(x)\; f_{i_r}(x) \prod_{b=r+1}^{n} \left\{ 1 - F_{i_b}(x) \right\} dx$$

$$= \sum_p \frac{1}{\theta_{i_r}} \int_0^T x^k \prod_{a=1}^{r-1} F_{i_a}(x) \prod_{b=r}^{n} \left\{ 1 - F_{i_b}(x) \right\} dx + \sum_p S_{i_r} \int_0^T x^k$$

$$\prod_{\substack{b=r\\ b\neq j}}^{n} \left\{1 - F_{i_b}(x)\right\} dx \; - \sum_{j=1}^{r-1} \int_0^T x^{k+1} \prod_{\substack{a=1\\ a\neq j}}^{r-1} F_{i_a}(x) \; f_{i_j}(x)$$

$$\prod_{b=r+1}^{n} \left\{1 - F_{i_b}(x)\right\} dx \Bigg] \; + \sum_{p} S_{i_r} \Bigg[\sum_{j=1}^{r-1} \int_0^T x^{k+1} \prod_{a=1}^{r-1} F_{i_a}(x) \; f_{i_j}(x)$$

$$\prod_{\substack{b=r+1\\ b\neq j}}^{n} \left\{1 - F_{i_b}(x)\right\} dx \; + \sum_{j=1}^{r-1} \int_0^T x^{k+1} \prod_{\substack{a=1\\ a\neq j}}^{r-1} F_{i_a}(x) \; f_{i_j}(x)$$

$$- \qquad \prod_{b=r+1}^{n} \left\{1 - F_{i_b}(x)\right\} dx \Bigg]$$

$$= \left[\sum_{i=1}^{n} 1/\theta_i \right] (r-1)! \; (n-r)! \; \mu_{r:n}^{(k+1)} \; - \; (r-2)! \; (r-1) \; (n-r)! \sum_{i=1}^{n} \frac{1}{\theta_i}$$

$$\mu_{r-1:n-1}^{[i](k+1)} \; + \; (r-1)! \; (n-r-1)! \; (n-r) \sum_{i=1}^{n} S_i \; \mu_{r:n-1}^{[i](k+1)}$$

$$- \; (r-2)! \; (r-1) \; (n-r)! \sum_{i=1}^{n} S_i \; \mu_{r-1:n-1}^{[i](k+1)} \qquad \ldots(3.6.8)$$

We get the Theorem by rewriting (3.6.8).

THEOREM 3.6.3: (Balakrishnan, 1994)

For $n \geq 2$ and $k = 0, 1, 2, \ldots,$

$$\mu_{n:n}^{(k+1)} = \frac{1}{\left(\sum_{i=1}^{n} 1/\theta_i \right)} \left\{ (k+1) \; \mu_{n:n}^{(k)} + \sum_{i=1}^{n} \left(\frac{1}{\theta_i} + S_i \right) \mu_{n-1:n-1}^{[i](k+1)} \right.$$

$$\left. - \left(\sum_{i=1}^{n} S_i \right) T^{k+1} \right\} , \qquad \ldots(3.6.9)$$

PROOF: From (1.9.1), let us consider for $n \geq 2$ and $k = 0, 1,$

2, ...,

$$(n-1)!\, \mu_{n:n}^{(k+1)} = \sum_{p} \int_0^T x^k \prod_{a=1}^{n-1} F_{i_a}(x)\, f_{i_n}(x)\, dx$$

$$= \sum_{p} \frac{1}{\theta_{i_n}} \int_0^T x^k \prod_{a=1}^{n-1} F_{i_a}(x) \prod_{b=r}^{n} \left\{1 - F_{i_n}(x)\right\} dx + \sum_{p} S_{i_n} \int_0^T x^k \prod_{a=1}^{n-1} F_{i_a}(x)\, dx$$

Using (3.6.3) and integrating by parts treting x^k for integration and rest of the integral for differentiation, we obtain

$$(k+1)\,(n-1)!\, \mu_{n:n}^{(k)} = \sum_{p} \frac{1}{\theta_{i_n}} \Bigg[-\sum_{j=1}^{n-1} \int_0^T x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{n-1} F_{i_a}(x)\, f_{i_j}(x) \left\{1 - F_{i_n}(x)\right\} dx + \int_0^T x^{k+1} \prod_{a=1}^{n-1} F_{i_a}(x)\, f_{i_n}(x)\, dx \Bigg] + \sum_{p} S_{i_n} \Bigg[T^{k+1} - \sum_{j=1}^{n-1} \int_0^T x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{r-1} \cdot F_{i_a}(x)\, f_{i_j}(x)\, dx \Bigg] \qquad \ldots(3.6.10)$$

Split the first set of integrals in the first term on the RHS of (3.6.10) into two through the term $1 - F_{i_n}(x)$, we obtain

$$(k+1)\,(n-1)!\, \mu_{n:n}^{(k)} = \sum_{p} \frac{1}{\theta_{i_n}} \Bigg[\sum_{j=1}^{n} \int_0^T x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{n} F_{i_a}(x)\, f_{i_j}(x)$$

$$- \sum_{j=1}^{n-1} \int_0^T x^{k+1} \prod_{\substack{a=1 \\ a\neq j}}^{n-1} F_{i_a}(x)\, f_{i_j}(x)\, dx + \sum_p S_{i_n} \left[T^{k+1} - \sum_{j=1}^{n-1} \int_0^T x^{k+1} \right.$$

$$\left. \prod_{\substack{a=1 \\ a\neq j}}^{n-1} F_{i_a}(x)\, f_{i_j}(x)\, dx \right] = \left(\sum_{i=1}^{n} 1/\theta_i \right) (n-1)!\, \mu_{n:n}^{(k+1)} - (n-2)!\,(n-1)$$

$$\sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{n-1:n-i}^{[i](k+i)} + (n-1)! \left(\sum_{i=1}^{n} S_i \right) T^k - (n-2)!\,(n-1)$$

$$\sum_{i=1}^{n} S_i\, \mu_{n-1:n-1}^{[i](k+1)} \qquad \ldots(3.6.11)$$

Thereby we get Theorem(3.6.3) by simply rearranging (3.6.11).

REMARK 3.6.1: The recurrence relations presented in the Theorem(3.6.1) to Theorem(3.6.3) will be able to compute all the single moments of all order statistics in a simple recursive manner for any specified value of θ_i ($i = 1, 2, \ldots, n$) and truncation point T.

REMARK 3.6.2: Theorem(3.6.1) and Theorem(3.6.2), for the case r=1, along with a general relation established by Balakrishnan(1988) which expresses $\mu_{r:n}(k)$ in terms of the k^{th} moment of the smallest order statistics in samples of size up to n, will also enable to compute all the single moments of all order Statistics in a simple recursive manner. Similarly we get for the largest order statistic.

REMARK 3.6.3: For the case when the variables are independent and identically distributed as standard right truncated exponential (

that is, $\theta_1 = \ldots = \theta_n = 1$), Theorems (3.6.1) to (3.6.3) reduce to

$$\mu_{1:1}^{(k+1)} = (k+1)\, \mu_{1:1}^{(k)} - S\, T^{k+1} \quad , \qquad k \ge 0 \qquad \ldots(3.6.12)$$

$$\mu_{r:n}^{(k+1)} = \frac{1}{n} \left\{ (k+1)\, \mu_{r:n}^{(k)} + n\, (S+1)\, \mu_{r-1:n-1}^{(k+1)} - n\, S\, \mu_{r:n-1}^{(k+1)} \right\}$$

$$1 \le r \le n-1, \quad k \ge 0 \qquad \ldots(3.6.13)$$

And $$\mu_{n:n}^{(k+1)} = \frac{1}{n} \left\{ (k+1)\, \mu_{n:n}^{(k)} + n\, (S+1)\, \mu_{n-1:n-1}^{(k+1)} - n\, S\, T^{k+1} \right\}$$

$$n \ge 2, \quad k \ge 0 \qquad \ldots(3.6.14)$$

where $S = \dfrac{e^{-T}}{1-e^{-T}}$ and $\mu_{0:n-1}^{(k+1)} = 0$ for $n \ge 2$.

Now consider the case when X_i's are independent and non-identically untruncated exponential distributed which means

$$S_i = \frac{e^{-T/\theta_i}}{\theta_i\, (1 - e^{-T/\theta_i})} \to 0$$

$$T\, S_i \to 0$$

For $i = 1, 2, \ldots, n$. Then Theorem (3.6.1) to (3.6.3) reduces to

For $n \ge 1$ and $k = 0, 1, 2$

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i} \right]} \mu_{1:n}^{(k)} \qquad \ldots(3.6.14)$$

For $2 < r \le n$ and $k = 0, 1, 2, \ldots$

$$\mu_{r:n}^{(k+1)} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i} \right]} \left\{ (k+1)\, \mu_{r:n}^{(k)} + \sum_{i=1}^{n} 1/\theta_i\, \mu_{r-1:n-1}^{[i](k+1)} \right\} \qquad \ldots(3.6.15)$$

3.7 RECUERENCE RELATIONS FOR PRODUCT MOMENTS IN RIGHT TRUNNCATED EXPONENTIAL DISTRIBUTION

THEOREM 3.7.1: (Balakrishnan, 1994)

with S_i as defined in (3.6.4),

$$\mu_{1,2:2} = \frac{1}{\left[\sum_{i=1}^{2} \frac{1}{\theta_i}\right]} \left\{ (\mu_{1:2} + \mu_{2:2}) - T \sum_{i=1}^{2} S_i \, \mu_{1:1}^{[i]} \right\} \qquad \ldots(3.7.1)$$

PROOF : From (1.9.2.), let us consider

$$\mu_{i:2} = E(\overline{X}_{1:2} \; X_{22}^{0}) = \sum_{p} \int_0^T \int_x^T x \, f_{i_1}(x) f_{i_2}(x) \; dy \, dx$$

so that

$$\mu_{1:2} = \sum_{p} \int_0^T x \, f_{i_1}(x) \; I(x) \, dx \qquad \ldots(3.7.2)$$

here

$$I(x) = \int_x^T f_{i_2}(x) \, dy = \frac{1}{\theta_{i_2}} \int_x^T \left\{ 1 - F_{i_2}(y) \right\} dy + S_{i_2} \int_x^T dy$$

using (3.6.3) and integrating by parts, we get

$$I(x) = \frac{1}{\theta_{i_2}} \left[\int_x^T y \, f_{i_2}(y) \, dy - x \left\{ 1 - F_{i_2}(y) \right\} \right] + S_{i_2} (T-x)$$

Now substitute this value in (3.7.2), which gives

$$\mu_{1:2} = \sum_{p} \frac{1}{\theta_{i_2}} \left[\int_0^T \int_x^T xy \, f_{i_2}(x) \, f_{i_2}(y) \, dy \, dx \right.$$

$$\left. - \int_0^T x^2 \, f_{i_2}(x) \left\{ 1 - F_{i_2}(x) \right\} dx \right]$$

$$+ \sum_{p} S_{i_2} \left[T \int_0^T x\, f_{i_1}(x)\, dx - \int_0^T x^2 f_{i_1}(x)\, dx \right] \qquad \ldots(3.7.3)$$

Next from (1.9.2), we consider

$$\mu_{2:2} = E\left(X^{0}_{1:2} \quad X_{2:2} \right)$$

$$= \sum_{p} \int_0^T \int_0^y y\, f_{i_1}(x)\, f_{i_2}(y)\, dx\, dy$$

so that

$$\mu_{2:2} = \sum_{p} \int_0^T y\, f_{i_2}(y)\, J(y)\, dy \qquad \ldots(3.7.4)$$

where

$$J(y) = \int_0^y f_{i_1}(x)\, dx$$

$$= \frac{1}{\theta_{i_1}} \int_0^y \left\{ 1 - F_{i_1}(x) \right\} dx + S_{i_1} \int_0^y dx$$

Using (3.6.3) and integrating by parts which yield

$$J(y) = \frac{1}{\theta_{i_1}} \left[y \left\{ 1 - F_{i_1}(y) \right\} + \int_0^y x\, f_{i_1}(x)\, dx \right] + S_{i_1} y$$

now substitute this value in (3.7.4)

$$\mu_{2:2} = \sum_{p} \frac{1}{\theta_{i_1}} \left[\int_0^T \int_x^T xy\, f_{i_1}(x)\, f_{i_2}(y)\, dy \quad dx \right.$$

$$\left. - \int_0^T y^2 \left\{ 1 - F_{i_1}(y) \right\} f_{i_2}(y)\, dy \right] + \sum_{p} S_{i_1} \int_0^T y^2 f_{i_2}(y)\, dy \qquad \ldots(3.7.5)$$

On adding (3.7.3) and (3.7.5), we get

$$\mu_{1:2} + \mu_{2:2} = \left(\sum_{i=1}^{2} 1/\theta_i \right) \mu_{1,2:2} + T \sum_{i=1}^{2} S_i \, \mu_{1:1}^{[i]}$$

which immediately yields the Theorem 3.7.1..

THEOREM 3.7.2: (Balakrishnan, 1994)

For $1 \le r \le n-2$,

$$\mu_{r,r+1:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i} \right]} \left\{ (\mu_{r:n} + \mu_{r+1:n}) + \sum_{i=1}^{n} (1/\theta_i + S_i) \, \mu_{r-1,r:n-1}^{[i]} - \sum_{i=1}^{n} S_i \, \mu_{r,r+1:n-1} \right\}, \quad \ldots(3.7.6)$$

with convention that $\mu_{0,1:n-1}^{[i]} = 0$ for $n \ge 3$.

PROOF: From (1.9.2), let us consider for $1 \le r \le n-2$

$$(r-1)!(n-r-1)! \, \mu_{r:n} = (r-1)!(n-r-1)! \, E(X_{r:n} \, X_{r+1:n}^{0})$$

$$= \sum_{p} \int_0^T \int_x^T x \prod_{a=1}^{r-1} f_{i_a}(x) \, f_{i_r}(x) \, f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dx\, dy$$

$$= \sum_{p} \int_0^T x \prod_{a=1}^{r-1} F_{i_a}(x) \, f_{i_r}(x) \, I(x) \, dx \quad \ldots(3.7.7)$$

where

$$I(x) = \int_x^T f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$= \frac{1}{\theta_{i_{r+1}}} \int_x^T \prod_{b=r+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy + S_{i_{r+1}} \int_x^T \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

using (3.6.3) and integerating by parts which yields

$$I(x) = \frac{1}{\theta_{i_{r+1}}} \left[\sum_{j=r+1}^{n} \int_{x}^{T} y\, f_{i_j}(y) \prod_{\substack{b=r+1 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy - x \prod_{b=r+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} \right]$$

$$+ S_{i_{r+1}} \left[\sum_{j=r+2}^{n} \int_{x}^{T} y\, f_{i_j}(y) \prod_{\substack{b=r+1 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy - \underline{x} \prod_{b=r+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} \right]$$

Put the above value of I(x) in (3.7.7), we get

$$(r-1)!(n-r-1)!\, \mu_{r:n} = \sum_{p} \frac{1}{\theta_{i_{r+1}}}$$

$$\left[\sum_{j=r+1}^{n} \int_{0}^{T} \int_{x}^{T} xy \prod_{a=1}^{r-1} F_{i_a}(x)\, f_{i_r}(x)\, f_{i_j}(y) \prod_{\substack{b=r+1 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\, dx \right.$$

$$\left. - \int_{0}^{T} x^2 \prod_{a=1}^{r-1} F_{i_a}(x)\, f_{i_r}(x) \prod_{b=r+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dx \right]$$

$$+ \sum_{p} S_{i_{r+1}} \left[\sum_{j=r+2}^{n} \int_{0}^{T} \int_{x}^{T} xy \prod_{a=1}^{r-1} F_{i_a}(x)\, f_{i_r}(x)\, f_{i_j}(y) \right.$$

$$\prod_{\substack{b=r+2 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\, dx$$

$$- \int_0^T x^2 \prod_{a=1}^{r-1} F_{i_a}(x)\ f_{i_r}(x) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dx$$
...(3.7.8)

from (1.9.2), let us write for $1 \le r \le n-2$

$$(r-1)!(n-r-1)!\ \mu_{r+1:n} = (r-1)!(n-r-1)!\ E\ (\ X^0_{r:n} \quad X_{r+1:n}\ )$$

$$= \sum_p \int_0^T \int_0^y y \prod_{a=1}^{r-1} F_{i_a}(x)\ f_{i_r}(x)\ f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dx\ dy$$

$$= \sum_p \int_0^T y\ f_{i_{r+1}}(x) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} J(y)\ dy$$
...(3.7.9)

where

$$J(y) = \int_0^y \prod_{a=1}^{r-1} F_{i_a}(x)\ f_{i_r}(x)\ dx$$

$$= \frac{1}{\theta_{i_r}} \int_0^y \prod_{a=1}^{r-1} F_{i_a}(x) \left\{ 1 - F_{i_r}(x) \right\} dx \ + S_{i_r} \int_0^y \prod_{a=1}^{r-1} F_{i_a}(x)\ dx$$

using (3.6.3) and integrating by parts, we get

$$J(y) = \frac{1}{\theta_{i_r}} \Bigg[y \prod_{a=1}^{r-1} F_{i_a}(y) \left\{ 1 - F_{i_r}(y) \right\}$$

$$- \sum_{j=1}^{r-1} \int_0^y x \prod_{\substack{a=1 \\ a \ne j}}^{r-1} F_{i_a}(x)\ f_{i_j}(x) \left\{ 1 - F_{i_r}(x) \right\} dx$$

$$+ \int_0^y x \prod_{a=1}^{r-1} F_{i_a}(x)\ f_{i_r}(x)\ dx \Bigg]$$

$$+ \; S_{i_r} \left[y \prod_{a=1}^{r-1} F_{i_a}(y) \; - \; \sum_{j=1}^{r-1} \int_0^y x \prod_{\substack{a=1 \\ a \neq j}}^{r-1} F_{i_a}(x) \; f_{i_j}(x) \, dx \right]$$

upon substituting this exxpression of J(y) in (3.7.9), we get

$$(r-1)!(n-r-1)! \; \mu_{r+1:n} \; = \; \sum_p \frac{1}{\theta_{i_r}}$$

$$\Bigg[\int_0^T y^2 \prod_{a=1}^{r-1} F_{i_a}(x) \left\{ 1 - F_{i_r}(y) \right\} f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$- \sum_{j=1}^{r-1} \int_0^T \int_x^T xy \prod_{\substack{a=1 \\ a \neq j}}^{r-1} F_{i_a}(x) \; f_{i_j}(x) \left\{ 1 - F_{i_r}(x) \right\} f_{i_{r+1}}(y)$$

$$\prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \; dx$$

$$+ \int_0^T \int_x^T xy \prod_{a=1}^{r-1} F_{i_a}(x) \; f_{i_r}(x) \; f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \; dx \; \Bigg]$$

$$+ \sum_p S_{i_r} \Bigg[\int_0^T y^2 \prod_{a=1}^{r-1} F_{i_a}(y) \; f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$- \sum_{j=1}^{r-1} \int_0^T \int_x^T xy \prod_{\substack{a=1 \\ a \neq j}}^{r-1} F_{i_a}(x) \; f_{i_j}(x) \; f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \; dx \Bigg] \quad \ldots(3.7.10)$$

Adding (3.7.8) and (3.7.10) and simplifying the resulting expression, we get

$$(r-1)!(n-r-1)! \; (\mu_{r:n} + \mu_{r+1:n})$$

$$= \; \left(\sum_{i=1}^{n} 1/\theta_i \right) (r-1)!(n-r-1)! \; \mu_{r,r+1:n}$$

$$+ \quad (r-1)!(n-r-2)!\ (n-r-1) \sum_{i=1}^{n} S_i\ \mu^{[i]}_{r,r+1:n-1}$$

$$- \quad (r-2)!\ (r-1)\ (n-r-1)! \sum_{i=1}^{n} \frac{1}{\theta_i}\ \mu^{[i]}_{r-1,r:n-1}$$

$$- \quad (r-2)!\ (r-1)\ (n-r-1)! \sum_{i=1}^{n} S_i\ \mu^{[i]}_{r-1,r:n-1}$$

Rearranging the above expression yield the proof.

THEOREM 3.7.3: (Balakrishnan, 1994)

For $n \geq 3$,

$$\mu_{n-1,n:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{n-1:n} + \mu_{n:n}) + \sum_{i=1}^{n} (1/\theta_i + S_i)\ \mu^{[i]}_{n-2,n-1:n-1} - T \sum_{i=1}^{n} S_i\ \mu^{[i]}_{n-1:n-1} \right\}, \qquad \ldots(3.7.11)$$

where S_i is as define in (3.6.4).

PROOF: We can prove this theorem in similar fashion as proved in previous theorem.

THEOREM 3.7.4: (Balakrishnan, 1994)

For $n \geq 4$ and $3 \leq s \leq n-1$,

$$\mu_{1,s:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{1:n} + \mu_{s:n}) - \sum_{i=1}^{n} S_i\ \mu^{[i]}_{1,s:n-1} \right\} \qquad \ldots(3.7.12)$$

PROOF: From (1.9.2), let us consider for $3 \leq s \leq n-1$

$$(s-2)!(n-s)!\ \mu_{1:n} = (s-2)!(n-s)!\ E\ (X_{1:n}\ X^{0}_{s:n})$$

$$= \sum_{P} \int_{0}^{T} \int_{x}^{T} x\ f_{i_1}(x) \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_s}(y)$$

$$\prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} \; dy \; dx$$

$$= \sum_{p} \int_0^T x \; f_{i_1}(x) \; I(x) \; dx \qquad \ldots(3.7.13)$$

where

$$I(x) = \int_x^T \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_s}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \; dx$$

$$= \frac{1}{\theta_{i_s}} \int_x^T \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} \prod_{b=s}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$+ \; S_{i_s} \int_x^T \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

using (3.6.3) and integrating by parts yields

$$I(x) = \frac{1}{\theta_{i_s}} \Bigg[- \sum_{j=2}^{s-1} \int_x^T y \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y)$$

$$\prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$+ \sum_{j=s}^{n} \int_x^T y \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y) \prod_{\substack{b=s \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \Bigg]$$

$$+ \; S_{i_s} \Bigg[- \sum_{j=2}^{s-1} \int_x^T y \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y)$$

$$\prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy$$

$$+ \sum_{j=s+1}^{n} \int_x^T y \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y) \prod_{\substack{b=s+1 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \Bigg]$$

Above expression of I(x) substitute in (3.7.13), which gives

$$(s-2)!(n-s)!\ \mu_{1:n} = \sum_{p} \frac{1}{\theta_{i_s}} \left[- \sum_{j=2}^{s-1} \int_{0}^{T} \int_{x}^{T} xy\ f_{i_1}(x) \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y) \prod_{b=s}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\ dx \right.$$

$$+ \sum_{j=s}^{n} \int_{0}^{T} \int_{x}^{T} xy\ f_{i_1}(x) \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y)$$

$$\left. \prod_{\substack{b=s \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\ dx \right]$$

$$+ \sum_{p} S_{i_s} \left[- \sum_{j=2}^{s-1} \int_{0}^{T} \int_{x}^{T} xy\ f_{i_1}(x) \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\ dx \right.$$

$$+ \sum_{j=s+1}^{n} \int_{0}^{T} \int_{x}^{T} xy\ f_{i_1}(x) \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(y)$$

$$\left. \prod_{\substack{b=s+1 \\ b \neq j}}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\ dx \right] \qquad \ldots(3.7.14)$$

Next from (1.9.2) write for $3 \leq s \leq n-1$

$$(s-2)!(n-s)!\ \mu_{s:n} = (s-2)!(n-s)!\ E\,(X^{0}_{1:n}\ X_{s:n})$$

$$= \sum \int_{0}^{T} y\ f_{i_s}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} J(y)\ dy \qquad \ldots(3.7.15)$$

where

$$J(y) = \int_0^y f_{i_1}(y) \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} dx$$

$$= \frac{1}{\theta_{i_1}} \int_0^y \left\{ 1 - F_{i_1}(x) \right\} \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} dx$$

$$+ \; S_{i_1} \int_0^y \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} dx$$

using (3.6.3) and integerating by parts, we get

$$J(y) = \frac{1}{\theta_{i_1}} \left[\int_0^y x \, f_{i_1}(x) \prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} dx \right.$$

$$+ \sum_{j=2}^{s-1} \int_0^y x \left\{ 1 - F_{i_1}(x) \right\} \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(x) \, dx$$

$$+ \; S_{i_1} \sum_{j=2}^{s-1} \int_0^y x \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(x) \, dx$$

Above expression of J(y) substitute in (3.7.15), which gives

$$(s-2)!(n-s)! \, \mu_{s:n} = \sum_p \frac{1}{\theta_{i_1}} \left[\int_0^T \int_x^T xy \, f_{i_1}(x) \right.$$

$$\prod_{a=2}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_s}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \, dx$$

$$+ \sum_{j=2}^{s-1} \int_0^T \int_x^T xy \left\{ 1 - F_{i_1}(x) \right\} \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(x)$$

$$\left. f_{i_s}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy \, dx \right]$$

$$+ \sum_{p} S_{i_1} \sum_{j=2}^{s-1} \int_0^T \int_x^T xy \prod_{\substack{a=2 \\ a \neq j}}^{s-1} \left\{ F_{i_a}(y) - F_{i_a}(x) \right\} f_{i_j}(x)$$

$$f_{i_s}(y) \prod_{b=s+1}^{n} \left\{ 1 - F_{i_b}(y) \right\} dy\, dx \qquad \ldots(3.7.16)$$

Adding (3.7.14) and (3.7.16) and simplifying the resulting expression, we get

$$(s-2)!(n-s)! \left(\mu_{1:n} + \mu_{s:n} \right) = \left(\sum_{i=1}^{n} 1/\theta_i \right) (s-2)!(n-s)!\, \mu_{1,s:n}$$

$$+ \ (s-2)!\ (n-s)\ (n-s-1)!\ \sum_{i=1}^{n} S_i\, \mu^{[i]}_{1,s:n-1}$$

Rearranging the above expression yield the proof.

THEOREM 3.7.5: (Balakrishnan, 1994)

For $n \geq 3$

$$\mu_{1,n:n} = \frac{1}{\left(\sum^{n} \frac{1}{\theta_i} \right)} \left\{ \left(\mu_{1:n} + \mu_{n:n} \right) - T \sum_{i=1}^{n} S_i\, \mu^{[i]}_{1:n-1} \right\}, \qquad \ldots(3.7.17)$$

PROOF: We can prove this theorem in similar fashion as in previous theorem.

THEOREM 3.7.6: (Balakrishnan, 1994)

For $2 \leq r < s \leq n-1$ and $s-r \geq 2$,

$$\mu_{r,s:n} = \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{\theta_i} \right)} \left\{ \left(\mu_{r:n} + \mu_{s:n} \right) \right.$$

$$\left. + \sum_{i=1}^{n} \left(\frac{1}{\theta_i} + S_i \right) \mu^{[i]}_{r-1,s-1:n-1} - \sum_{i=1}^{n} S_i\, \mu^{[i]}_{r,s:n-1} \right\} \qquad \ldots(3.7.1\theta)$$

PROOF: Please see the proof of Theorem (3.7.3).

THEOREM 3.7.7: (Balakrishnan, 1994)

For $2 \le r \le n-2$,

$$\mu_{r,n:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{r:n} + \mu_{n:n}) \sum_{i=1}^{n} \left(\frac{1}{\theta_i} + S_i \right) \mu^{[i]}_{r-1,n-1:n-1} - T \sum_{i=1}^{n} S_i \, \mu^{[i]}_{r:n-1} \right\} \qquad \ldots(3.7.19)$$

REMARK 4.7.4: The recurrence relations presented by Theorems (3.7.1)to (3.7.7) will be enable one to compute all the product moments and hence the covariances of all order statistics in a simple recursive manner for any specified values of θ_i (i = 1, 2, ..., n) and the truncation point T.

REMARK 3.7.5: For the case when random variables independent nad nonidenticallly distributed, previous theorems reduceced as

$$(r-1)\, \mu_{r,s:n} + (s-r)\, \mu_{r-1,s:n} + (n-s+1)\, \mu_{r-1,s-1:n} = \sum_{i=1}^{n} \mu^{[i]}_{r-1,s-1:n} \qquad \ldots(3.7.20)$$

REMARK 3.7.6: Let us consider the case when X_i's are independent and nonidentically untruncated exponentially distributed, then

$$S_i = \frac{e^{-T/\theta_i}}{\theta_i \, (1 - e^{-T/\theta_i})} \to 0$$

$$T\, S_i \to 0$$

for i = 1, 2, ..., n. And relatios in previous theorems reduce as for $n \geq 2$,

$$\mu_{1,2:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{1:n} + \mu_{2:n}) \right\} \qquad \ldots(3.7.21)$$

for $2 \leq r \leq n-1$

$$\mu_{r,r+1:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{r:n} + \mu_{r+1:n}) + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1,r:n-1} \right\} \qquad \ldots(3.7.22)$$

for $3 \leq s \leq n$

$$\mu_{1,s:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{1:n} + \overline{\mu}_{s:n}) \right\} \qquad \ldots(3.7.23)$$

and for $2 \leq r < s \leq n$ and $s-r \geq 2$,

$$\mu_{r,s:n} = \frac{1}{\left[\sum_{i=1}^{n} \frac{1}{\theta_i}\right]} \left\{ (\mu_{r:n} + \mu_{s:n}) + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu^{[i]}_{r-1,s-1:n-1} \right\} \qquad \ldots(3.7.24)$$

3.8. RECURRENCE RELATIONS FOR p- OUTLIER IN RIGHT TRUNCATED EXPONENTIAL MODEL

We assume that $X_1, X_2, \ldots, X_{n-p}$ are independent right truncated exponential random variables with parameter θ, while $X_{n-p+1}, \ldots, X_n$ are independent right truncated exponential random variables with parameter T and they are independent of X_1, $X_2, \ldots, X_{n-p}$; see Barnett and Lewis (1994). Again let us suppose single moments by $\mu^{(k)}_{r:n}[p]$ and the product moments by $\mu_{r,s:n}[p]$ for p outliers model. Similarly we denote single moment $\mu^{(k)}_{r:n-1}[p-1]$ and product moment $\mu_{r,s:n-1}[p-1]$ for sample size n-1 consisting p-1 outliers.

IDENTITY 3.8.1: (Balakrishnan, 1994)

For $k = 0, 1, 2, \ldots,$

$$\mu_{1:1}^{(k+1)}[0] = \frac{1}{1/\theta} \left\{ (k+1)\, \mu_{1:n}^{(k)}[0] - S_\theta\, T^{k+1} \right\} \qquad \ldots(3.8.1)$$

IDENTITY 3.8.2: (Balakrishnan, 1994)

For $1 \le r \le n-1$ and $k = 0, 1, 2, \ldots,$

$$\mu_{r:n}^{(k+1)}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T} \right)} \left\{ (k+1)\, \mu_{r:n}^{(k)}[p] + (n-p) \left(\frac{1}{\theta} - S_\theta \right) \mu_{r-1:n-1}^{(k+1)}[p] + p \left(\frac{1}{T} + S_T \right) \mu_{r-1:n-1}^{(k+1)}[p-1] - (n-p)\, S_\theta\, \mu_{r:n-1}^{(k+1)}[p] - p\, S_T\, \mu_{r:n-1}^{(k+1)}[p-1] \right\} \qquad \ldots(3.8.2)$$

IDENTITY 3.8.3: (Balakrishnan, 1994)

For $n \ge 2$ and $k = 0, 1, 2, \ldots,$

$$\mu_{n:n}^{(k+1)}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T} \right)} \left\{ (k+1)\, \mu_{n:n}^{(k)}[p] + (n-p) \left(\frac{1}{\theta} - S_\theta \right) \mu_{n-1:n-1}^{(k+1)}[p] + p \left(\frac{1}{T} + S_T \right) \mu_{n-1:n-1}^{(k+1)}[p-1] - \left\{ (n-p)\, S_\theta + p\, S_T \right\} T^{k+1} \right] \qquad \ldots(3.8.3)$$

IDENTITY 3.8.4: (Balakrishnan, 1994)

$$\mu_{1,2:2}[0] = \frac{1}{2/\theta} \left\{ \left(\mu_{1:2}[0] + \mu_{2:2}[0] \right) - 2T\, S_\theta\, \mu_{1:1}[0] \right\} \qquad \ldots(3.8.4)$$

IDENTITY 3.8.5: (Balakrishnan, 1994)

For $1 \le r \le n-2$

$$\mu_{r,r+1:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T}\right)} \Big\{ (\mu_{r:n}[p] + \mu_{r+1:n}[p])$$

$$+ (n-p)\left(\frac{1}{\theta} + S_\theta\right) \mu_{r-1,r:n-1}[p] + p\left(\frac{1}{T} + S_T\right)\mu_{r-1,r:n-1}[p-1]$$

$$- (n-p)\, S_\theta\, \mu_{r,r+1:n-1}[p] - p\, S_T\, \mu_{r,r+1:n-1}[p-1] \Big\} \quad \ldots(3.8.5)$$

IDENTITY 3.8.6: (Balakrishnan, 1994)

For $n \geq 3$

$$\mu_{n-1,n:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T}\right)} \Big\{ (\mu_{n-1:n}[p] + \mu_{n:n}[p])$$

$$+ (n-p)\left(\frac{1}{\theta} - S_\theta\right) \mu_{n-2,n-1:n-1}[p] + p\left(\frac{1}{T} + S_T\right)\mu_{n-2,n-1:n-1}[p-1]$$

$$- (n-p)\, S_\theta\, T\, \mu_{n-1:n-1}[p] - p\, S_T\, T\, \mu_{n-1:n-1}[p-1] \Big\} \quad \ldots(3.8.6)$$

IDENTITY 3.8.7: (Balakrishnan, 1994)

For $3 \leq s \leq n-1$

$$\mu_{1,s:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T}\right)} \Big\{ (\mu_{1:n}[p] + \mu_{s:n}[p])$$

$$- (n-p)\, S_\theta\, \mu_{1,s:n-1}[p] - p\, S_T\, \mu_{1,s:n-1}[p-1] \Big\} \quad \ldots(3.8.7)$$

IDENTITY 3.8.8: (Balakrishnan, 1994)

For $n \geq 3$

$$\mu_{1,n:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T}\right)} \Big\{ (\mu_{1:n}[p] + \mu_{n:n}[p])$$

$$- (n-p)\, S_\theta\, T\, \mu_{1:n-1}[p] - p\, S_T\, T\, \mu_{1:n-1}[p-1] \Big\} \quad \ldots(3.8.8)$$

IDENTITY 3.8.9: (Balakrishnan, 1994)

For $2 \le r < s \le n-1$ and $s-r \ge 2$,

$$\mu_{r,s:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T} \right)} \Big\{ (\mu_{r:n}[p] + \mu_{s:n}[p])$$

$$+ (n-p)\left(\frac{1}{\theta} - S_\theta \right) \mu_{r-1,s-1:n-1}[p] + p\left(\frac{1}{T} + S_T \right)\mu_{r-1,s-1:n-1}[p-1]$$

$$- (n-p)\, S_\theta\, T\, \mu_{r,s:n-1}[p] - p\, S_T\, \mu_{r,s:n-1}[p-1] \Big\} \quad \ldots(3.8.9)$$

IDENTITY 3.8.10: (Balakrishnan, 1994)

For $2 \le r \le n-2$

$$\mu_{r,n:n}[p] = \frac{1}{\left(\frac{n-P}{\theta} + \frac{P}{T} \right)} \Big\{ (\mu_{r:n}[p] + \mu_{n:n}[p])$$

$$+ (n-p)\left(\frac{1}{\theta} - S_\theta \right) \mu_{r-1,n-1:n-1}[p] + p\left(\frac{1}{T} + S_T \right)\mu_{r-1,n-1:n-1}[p-1]$$

$$- (n-p)\, S_\theta\, T\, \mu_{r:n-1}[p] - P\, S_T\, \mu_{r:n-1}[p-1] \Big\} \quad \ldots(3.8.10)$$

where

$$S_\theta = \frac{1}{\theta} \cdot \frac{e^{-T/\theta}}{1 - e^{-T/\theta}}$$

$$S_\tau = \frac{1}{\tau} \cdot \frac{e^{-T/\tau}}{1- e^{-T/\tau}}$$

and τ denote truncation point.

Chapter-IV

RECURRENCE RELATIONS FOR NONINDEPENDENT NONIDENTICAL RANDOM VARIABLES

4.1 INTRODUCTION

Balakrishnan(1988) derived recurrence relations when random variables are independent and nonidentically distributed. Sathe and Dixit(1990) established recurrence relations and identities for order statistics for the randor variables assumed to be nonindependent nonidentically distributed. The recurrence relations are given by Sathe and Dixit(1990) are

$$r\, F_{r+1,n}(x) \;+\; (n-r)\, F_{r,n}(x) \;=\; \sum_{i=1}^{n} F^{(i)}_{r,n-1}(x),$$

$$1 \le r \le n-1 \qquad \ldots(4.1.1)$$

and

$$r\, F_{r+1,s+1,n}(x,y) \;+\; (s-r)\, F_{r,s+1n}(x,y) \;+\; (n-s)\, F_{r,s,n}(x,y)$$

$$= \sum_{i=1}^{n} F^{(i)}_{r,s,n-1}(x),$$

$$1 \le r < s \le n-1 \qquad \ldots(4.1.2)$$

where $F^{(i)}_{r,n-1}(x)$ and $F^{(i)}_{r,s,n-1}(x,y)$denote distribution function of $X_{r,n-1}$ and $(X_{r,n-1}, X_{s,n-1})$ in sample of size n-1 obtained on dropping X_i from the original sample of size n.

Balakrishnan(1992) established some new identities and generalized some of the established results to the nonindependent nonidentically distributed random varible case. These results greatly reduce the amount of direct computations when the random variables are not necessarily i.i.d.. To start with we state and prove a lemma before stating the main results.

LEMMA 4.1.1: (Balakrishnan *et al.* 1992)

For real positive k and c and a positive integer b,

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} \beta(a+k,c) = \beta(k,c+b)$$

where $\beta(.,.)$ is a beta function.

PROOF: Consider

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} \beta(a+k,c) = \sum_{a=0}^{b} (-1)^a \binom{b}{a} \int_0^1 u^{a+k}(1-u)^{c-1}\, du$$

on changing the order of summation and integeration, we get

$$= \int_0^1 \left\{ \sum_{a=0}^{b} (-1)^a \binom{b}{a} u^a \right\} u^{k-1} (1-u)^{c-1}\, du$$

thereby prove the lemma.

REMARK 4.1.1: The lemma is true for incomplete beta itegrals in general and we get

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} I_p(a+k,c) = I_p(k,c+b)$$

where $I_p(a,b)$ is defined as the incomplete beta integral given by

$$I_p(a,b) = \int_0^P u^{a-1}(1-u)^{b-1}\, du\ , \qquad P \in (0,1) .$$

Let $F_{r,m}^{[i_1, \ldots, i_{n-m}]}(x)$, $1 \le r \le m \le n$ denote the distribution function of r^{th} order statistics in the sample of size m obtained on dropping $X_{i_1}, X_{i_2}, \ldots, X_{i_{n-m}}$ from original sample of size n. Further suppose

$$H_{r,m}(x) = \sum_{1 \le i_1 < i_2 \ldots < i_{n-m} \le n} F_{r,m}^{[i_1, \ldots, i_{n-m}]}(x).$$

For $m = n$, $H_{r,n}(x) = F_{r,n}(x)$, $1 \le r \le n$.

And when the variables are identically distributed

$$H_{r,m}(x) = \binom{n}{m} F_{r,m}(x) .$$

4.2 RESULTS FOR NONINDEPENDENT NONIDENTICALLY DISTRIBUTED RANDOM VARIABLES

RESULT 4.2.1: (Balakrishnan *et al.* 1992)

For $1 \le r \le n$

$$F_{r,n}(x) = \sum_{j=n-r+1}^{n} (-1)^{j+n-r+1} \binom{j-1}{n-r} H_{1,j}(x) \qquad \ldots(4.2.1)$$

$$F_{r,n}(x) = \sum_{j=r}^{n} (-1)^{j+r} \binom{j-1}{r-1} H_{j,j}(x) \qquad \ldots(4.2.2)$$

PROOF: From Lemma(4.1.1), we have

$$F_{r,n}(x) = \frac{n-r+1}{r-1} F_{r-1,n}(x) + \frac{1}{r-1} \sum_{i_1=1}^{n} F_{r-1,n-1}^{[i_1]}(x)$$

upon using (4.1.1) to the RHS of the above equation , we get

$$F_{r,n}(x) = \frac{(n-r+2)(n-r+1)}{(r-1)(r-2)} F_{r-2,n}(x) - 2 \frac{(n-r+1)}{(r-1)(r-2)}$$

$$\sum_{i_1=1}^{n} F_{r-2,n-1}^{[i_1]}(x) + \frac{1}{(r-1)(r-2)} \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \\ i_1 \neq i_2}}^{n} F_{r-1,n-1}^{[i_1,i_2]}(x) .$$

Now repeatthe process of using (4.1.1) for the expression on the RHS (r-1) times and simplifying the resulting equation, we derive the Result(4.2.1)

RESULT 4.2.2: (Balakrishnan *et al.* 1992)

$$\sum_{r=1}^{n} \frac{1}{r} F_{r,n}(x) = \sum_{r=1}^{n} \beta(r,n-r+1)\, H_{1,r}(x) \qquad \ldots(4.2.3)$$

$$\sum_{r=1}^{n} \frac{1}{n-r+1} F_{r,n}(x) = \sum_{r=1}^{n} \beta(r,n-r+1)\, H_{r,r}(x) \qquad \ldots(4.2.4)$$

PROOF: From Result (4.2.1), we have

$$\sum_{r=1}^{n} \frac{1}{r} F_{r,n}(x) = \sum_{r=1}^{n} \frac{1}{r} \sum_{j=n-r+1}^{n} (-1)^{j+n-r+1} \binom{j-1}{n-r} H_{1,j}(x)$$

on interchanging the order of summation and making transformation, the RHS reduce to

$$= \sum_{j=1}^{n} \left\{ \sum_{l=0}^{j-1} (-1)^{l} \binom{j-1}{l} \frac{1}{(n-j+l-1)} \right\} H_{1,j}(x)$$

from lemma(4.1.1) the term inside braces is $\beta(j,n-j+1)$ thereby establish Result(4.2.2).

For $i = 1, 2, \ldots,$ define for a fixed n

$$C_{i+k-1} = \begin{cases} (n+i)(n+i+1) \ldots (n+i+k-2), & k = 2, 3, \ldots \\ 1 & k = 1 \end{cases} \qquad \ldots(4.2.5)$$

RESULT 4.2.3: (Balakrishnan *et al.* 1992)

For $i, k = 1, 2, \ldots,$

$$\sum_{r=1}^{n} F_{r,n}(x) \Big/ \Big\{ (r+i-1)(r+i) \ldots (r+i+k-2) \Big\}$$

$$= \frac{1}{C_{i+k-1}} \sum_{r=1}^{n} \binom{k+r-2}{k-1} \beta(r, n-r+i) \, H_{1,r}(x) \qquad \ldots(4.2.6)$$

$$\sum_{r=1}^{n} F_{r,n}(x) \Big/ \Big\{ (n-r+i)(n-r+i+1) \ \ldots (n-r+i+k-1) \Big\}$$

$$= \frac{1}{C_{i+k-1}} \sum_{r=1}^{n} \binom{k+r-2}{k-1} \beta(r, n-r+i) \, H_{r,r}(x) \qquad \ldots(4.2.7)$$

RESULT 4.2.4: (Balakrishnan *et al.* 1992)

For $k = 1, 2, \ldots,$

$$\sum_{r=1}^{n} F_{r,n}(x) \Big/ \Big\{ r\,(r+1) \ldots (r+k-1)(n-r+1) \ldots (n-r+k) \Big\}$$

$$= \frac{1}{C_{2k}} \sum_{r=1}^{n} \binom{r+2k-2}{r-1} \beta(r, n-r+1) \Big\{ H_{1,r}(x) + H_{r,r}(x) \Big\} \qquad \ldots(4.2.8)$$

and for $k, l = 1, 2, \ldots,$

$$\sum_{r=1}^{n} F_{r,n}(x) \Big/ \Big\{ r\,(r+1) \ldots (r+k-1)(n-r+1) \ldots (n-r+l) \Big\}$$

$$= \frac{1}{C_{k+l}} \sum_{r=1}^{n} \beta(r, n-r+1) \Big\{ \binom{r+k+l-2}{k-1} H_{1,r}(x) + \binom{r+k+l-2}{l-1} H_{r,r}(x) \Big\}$$

where C_{2k} and C_{k+l} are defined in (4.2.5) with $i = 1$.

Let $F^{[i_1, \ldots, i_{n-m}]}_{r,s,m}(x,y)$, $1 \le r < s \le m \le n$ denote the joint distribution function of the r^{th} and s^{th} order statistics from a sample of size m , obtained on dropping $X_{i_1}, X_{i_2}, \ldots, X_{i_{n-m}}$ from original sample of size n. Again suppose

$$H_{r,s,m}(x,y) = \sum_{1 \le i_1 < i_2 \ldots < i_{n-m} \le n} F_{r,s,m}^{[i_1, \ldots, i_{n-m}]}(x,y).$$

For $m = n$, $H_{r,s,n}(x,y) = F_{r,s,n}(x,y)$, $1 \le r < s \le n.$

RESULT 4.2.5: (Balakrishnan *et al.* 1992)

For $1 \le r < s \le n.$

$$F_{r,s,n}(x,y) = \sum_{j=r}^{s-1} \sum_{m=n-s+j+1}^{n} (-1)^{m+n-r-s+1} \binom{j-1}{r-1} \binom{m-j-1}{n-s} H_{j,j+1,m}(x,y) \qquad \ldots(4.2.9)$$

$$F_{r,s,n}(x,y) = \sum_{j=s-r}^{s-1} \sum_{m=n-s+j+1}^{n} (-1)^{n-m-r+1} \binom{j-1}{s-r-1} \binom{m-j-1}{n-s} H_{1,j+1,m}(x,y) \qquad \ldots(4.2.10)$$

$$F_{r,s,n}(x,y) = \sum_{j=s-r}^{n-r} \sum_{m=r+j}^{n} (-1)^{m+s} \binom{j-1}{s-r-1} \binom{m-j-1}{r-1} H_{m-j,m,m}(x,y) \qquad \ldots(4.2.11)$$

PROOF: Above three idetities are proved by recurrence relation given by Result(4.2.2).

RESULT 4.2.6: (Balakrishnan *et al.* 1992)

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s-r}F_{r,s,n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\beta(s-1,n-s+1)\,H_{r,r+1,s}(x,y) \qquad \ldots(4.2.12)$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{r}F_{r,s,n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\beta(s-1,n-s+1)\,H_{1,r+1,s}(x,y) \qquad \ldots(4.2.13)$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{n-s+1}F_{r,s,n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\beta(s-1,n-s+1)\,H_{r,s,s}(x,y) \qquad \ldots(4.2.14)$$

PROOF: Proof of the above identities are identical as in Result(4.2.2).

Chapter - V

RECURRENCE RELATIONS FOR TWO RELATED MODELS WITH ONE OUTLIER

5.1 INRODUCTION

Govindarajulu(1963) derived recurrence relations among moments of order statistics in samples from two related populations for symmetric distributions. Khan and Khan(1986) extended to the truncated exponential and double exponential model for i.i.d. case. Balakrishnan and Ambagaspitiya(1988) established recurrence relations for two related symmetric outlier models. Furtner, Balakrishnan(1989b) derived the generalized case when the order statistics arise from two related sets of independent and nonidentical distributed random variables. These relations can beemployed to simplify the evaluation of the moments of order statistics in the symmetric outlier models. Govindarajulu *et al.* (1993) established the probablistic proof of recurrence relations for the independent and nonidentical distributed random varibles.

We have defined pdf and cdf, covariance and single and product moments of simple order statistics and in the presence of an outlier in (1.5.2),(1.5.4),(1.6.1) and (1.6.2).

To devlope the recurrence relations for symmetric outlier model, we make the following assumptions given by Khan *et al.*(1986) and Balakrishnan *et al.*(1988).

Let for $x > 0$,

$$\left.\begin{aligned} F^{*}(x) &= 2\,F(x) - 1 \\ f^{*}(x) &= 2\,f(x) \end{aligned}\right\} \qquad \ldots(5.1.1)$$

and

$$\left.\begin{aligned} G^{*}(x) &= 2\,G(x) - 1 \\ g^{*}(x) &= 2\,g(x) \end{aligned}\right\} \qquad \ldots(5.1.2)$$

where $f(x)$ and $g(x)$ are the pdf of order statistics in the presence of no outlier and in the presence of an outlier respectively. The density functions $f^{*}(x)$ and $g^{*}(x)$ are obtained by folding the density functions $f(x)$ and $g(x)$ at zero respectively. Again assume that the single and the product moments of order statistics in a random sample of size n drawn from a population with pdf $f^{*}(x)$and cdf $F^{*}(x)$ by $\nu^{*}_{r:n}(k)(1 \le r \le n)$ and $\nu^{*}_{r,s:n}(1 \le r < s \le n)$ respectively. Further we suppose single and product moments of order statistics obtained from a sample of n independent random variables out of which (n−1) have pdf $f^{*}(x)$ and cdf $F^{*}(x)$and one variable has pdf $g^{*}(x)$ and cdf $G^{*}(x)$ by $\mu^{*(k)}_{r:n}(1 \le r \le n)$ and $\mu^{*(k)}_{r,s:n}(1 \le r < s \le n)$ respectively.

5.2 RELATIONS AMONG MOMENTS OF ORDER STATISTICS

We present the relations established recently by Balakrishnan(1987,1990) which expres the moments $\mu^{(k)}_{r:n}$ $(1 \le r \le n)$ and $\mu_{r,s:n}$ in the terms of the moments $\mu^{*(k)}_{r:n}$, $\nu^{*(k)}_{r:n}$ $(1 \le r \le n)$ and $\mu^{*(k)}_{r,s:n}$, $\nu^{*(k)}_{r,s:n}$ $(1 \le r < s \le n)$.

RELATION 5.2.1: (Balakrishnan and Ambagaspitiya, 1988)

For $1 \le r \le n$ and $k = 1, 2, \ldots,$

$$\mu_{r:n}^{(k)} = 2^{-n} \Bigg[\sum_{i=1}^{r-1} \binom{n-1}{i-1} \nu_{r-i:n-i}^{*(k)} + (-1)^k \sum_{i=r}^{n-1} \binom{n-1}{i} \nu_{i-r+1:i}^{*(k)} + \sum_{i=1}^{r-1} \binom{n-1}{i} \mu_{r-i:n-i}^{*(k)} + \sum_{i=r}^{n} \binom{n-i}{i-1} \mu_{i-r+1:i}^{*(k)} \Bigg] \quad \ldots(5.2.1)$$

RELATION 5.2.2: *(Balakrishnan and Ambagaspitiya, 1988)*

For $1 \le r < s \le n$

$$2^n \mu_{r,s:n} = \sum_{i=1}^{r-1} \binom{n-1}{i-1} \nu_{r-i,s-i:n-i}^{*} + \sum_{i=r}^{n-1} \binom{n-1}{i} \nu_{i+1-s,i+1-r:i}^{*} + \sum_{i=0}^{r-1} \binom{n-1}{i} \mu_{r-i,s-i:n-i}^{*} + \sum_{i=s}^{n} \binom{n-1}{i-1} \mu_{i+1-s,i+1-r:i}^{*} - \sum_{i=r}^{s-1} \binom{n-1}{i-1} \nu_{s-i:n-i}^{*} \mu_{i+1-r:i}^{*} - \sum_{i=r}^{s-1} \binom{n-1}{i} \nu_{i+1-r:i}^{*} \mu_{s-i:n-i}^{*} \quad \ldots(5.2.2)$$

REMARK 5.2.1: If the moments $\mu_{r:m}^{*(k)}$, $\nu_{r:m}^{*(k)}$, $\mu_{r,s:m}^{*(k)}$ and $\nu_{r,s:n}^{*(k)}$ are all available for sample sizes upto n, then all the single moments $\mu_{r:n}^{(k)}$ $(1 \le r \le n)$ and the product moments $\mu_{r,s:n}$ $(1 \le r < s \le n)$ of order statistics in a sample of size n from a symmetric outlier model, with a single outlier, can be obtained by using Relations (5.2.1) & (5.2.2). Thus, for example, given the single and product moments of order statistics from standard exponential distribution and also the single and product moments of order statistics from the single scale outlier exponential model, the single and the product moments of order statistics from the single scale outlier double exponential model can all be obtained by using Relations (5.2.1) & (5.2.2).

REMARK 5.2.2: If $G(x) \equiv F(x)$

It means variable X is not an outlier then Relations (5.2.1) & (5.2.2) reduce as

$$\nu^{(k)}_{r:n} = 2^{-n} \left[\sum_{i=0}^{r-1} \binom{n}{i} \nu^{*(k)}_{r-i:n-i} + (-1)^k \sum_{i=r}^{n} \binom{n}{i} \nu^{*(k)}_{i-r+1:i} \right] \qquad \ldots(5.2.3)$$

and

$$2^n \nu_{r,s:n} = \sum_{i=0}^{r-1} \binom{n}{i} \nu^{*}_{r-i,s-i:n-i} + \sum_{i=s}^{n} \binom{n}{i} \nu^{*}_{i+1-s,i+1-r:i} - \sum_{i=0}^{s-1} \binom{n}{i} \nu^{*}_{i+1-r:i} \nu^{*}_{s-i:n-i} \qquad \ldots(5.2.4)$$

5.3 RECURRENCE RELATIONS FOR NON I.I.D. RANDOM VARIABLES

Let us assume that the density functions $f_i(x)$ are all symmetetric about zero. For $x > 0$, Let

$$\left.\begin{aligned} G_i(x) &= 2\,F_i(x) - 1 \\ g_i(x) &= 2\,f_i(x) \end{aligned}\right\} \qquad \ldots(5.3.1)$$

Thereby, the density functions $g_i(x)$, $i = 1, 2, \ldots, n$, are obtained by folding the density functions $f_i(x)$ at zero. Now assume $Y_{1:n} \le Y_{2:n} \le \ldots \le Y_{n:n}$ denote the order statistics obtained from n independent, absolutely continuous random variables Y_i ($i = 1, 2, \ldots, n$), with Y_i having pdf $g_i(x)$ and cdf $G_i(x)$. Let us denote $\nu_{r:n-1}^{(k)[i_1,\ldots,i_l]}$ for the k^{th} single moment of $Y_{r:n-1}^{[i_1,\ldots,i_l]}$ and $\nu_{r,s:n-1}^{[i_1,\ldots,i_l]}$ for the product moment of $Y_{r:n-1}^{[i_1,\ldots,i_l]}$ and $Y_{s:n-1}^{[i_1,\ldots,i_l]}$. Here $Y_{r:n-1}^{[i_1,\ldots,i_l]}$ denotes the r^{th} order statistics in a sample of size n−1 obtained by dropping Y_{i_1}, Y_{i_2}, ..., Y_{i_l} from original set of n variables Y_1, Y_2, .., Y_n.

RELATION 5.3.1: (Balakrishnan, 1989b)

For $1 \le r \le n$ and $k = 1, 2, \ldots,$

$$\mu_{r:n}^{(k)} = 2^{-n} \left[\sum_{l=0}^{r-1} \sum_{1 \le i_1 < \ldots i_l \le n} \nu_{r-l:n-l}^{(k)[i_1,\ \ldots,\ i_l]} \right.$$

$$\left. + (-1)^k \sum_{l=r}^{r-1} \sum_{1 \le i_1 < \ldots i_{n-l} \le n} \nu_{l-r+1:l}^{(k)[i_1,\ \ldots,\ i_l]} \right] \qquad \ldots(5.3.2)$$

PROOF: From (1.9.3) and (5.3.1), we have

$$\mu_{r:n}^{(k)} = \frac{2^{-n}}{(r-1)!(n-r)!} \left[\int_0^\infty x^k\, I_{r-1,n-r}(x)\, dx \right.$$

$$\left. + (-1)^k \int_0^\infty x^k\, I_{n-r,r-1}(x)\, dx \right], \qquad \ldots(5.3.3)$$

where

$$I_{r-1,n-r}(x) = \begin{vmatrix} 1+G_1(x) & 1+G_2(x) & \ldots & 1+G_n(x) \\ g_1(x) & g_2(x) & \ldots & g_n(x) \\ & & & \\ 1-G_1(x) & 1-G_2(x) & \ldots & 1-G_n(x) \end{vmatrix}^{+}_{+} \begin{array}{l} (r-1)\ \text{rows} \\ \\ \\ (n-r)\ \text{rows} \end{array}$$

Similarly get the expression for $I_{n-r,r-1}(x)$. Now using the cauchy expression of permanent (Aitkon (1944), P.74), We have

$$I_{r-1,n-r}(x) = \left|\begin{array}{cccc} 1+G_2(x) & 1+G_3(x) & \cdots & 1+G_n(x) \\ g_2(x) & g_3(x) & \cdots & g_n(x) \\ & & & \\ 1-G_2(x) & 1-G_3(x) & \cdots & 1-G_n(x) \end{array}\right|^{+} \begin{array}{l} (r-2) \text{ rows} \\ \\ \\ (n-r) \text{ rows} \end{array}$$

$$+ \left|\begin{array}{cccc} G_1(x) & 1+G_2(x) & \cdots & 1+G_n(x) \\ 1+G_1(x) & 1+G_2(x) & \cdots & 1+G_n(x) \\ g_1(x) & g_2(x) & \cdots & g_n(x) \\ & & & \\ 1-G_1(x) & 1-G_2(x) & \cdots & 1-G_n(x) \end{array}\right|^{+} \begin{array}{l} \\ (r-2) \text{ rows} \\ \\ \\ (n-r) \text{ rows} \end{array}$$

By repeating application, we get

$$I_{r-1,n-r}(x) = \sum_{i_1=1}^{n} J^{[i_1]}_{0,r-2,n-r}(x) + J_{1,r-2,n-r}(x),$$

where $J^{[i_1]}_{0,r-2,n-r}(x)$ is the permanent obtained from $I_{r-1,n-r}(x)$ by dropping the first row and i_1^{th} column and $J_{1,r-2,n-r}(x)$ is the permanent obtained from $I_{r-1,n-r}(x)$ by replacing the first row by $G_1(x)$, $G_2(x)$, ..., $G_n(x)$. Similarly we obtain

$$I_{r-1,n-r}(x) = \sum_{l=0}^{r-1} (r-l-1)! \binom{r-1}{l}$$

$$\sum_{1 \le i_1 \dots < i_{r-2-1} \le n} J_{1,0,n-r}^{[i_1, \dots, i_{r-1-1}]}(x)$$

where

$$J_{1,0,n-r}^{[i_1,\dots,i_{r-1-1}]} = \left|\begin{array}{cccc} G_1(x) & G_2(x) & \dots & G_n(x) \\ g_1(x) & g_2(x) & \dots & g_n(x) \\ & & & \\ 1 - G_1(x) & 1 - G_2(x) & \dots & 1 - G_n(x) \end{array}\right| \begin{array}{l} 1 \\ \text{rows} \\ \\ n-r \\ \text{rows} \end{array}$$

with columns $(i_1, i_2, \dots, i_{r-1-1})$ have been dropped and by realizing that —

$$\int_0^{\infty} x^k J_{1,0,n-r}^{[i_1,\dots,i_{r-1-1}]}(x)\, dx = 1!\,(n-r)!\, \nu_{1+1:n-r+1+1}^{(k)[i_1,\dots,i_{r-1-1}]}$$

and

$$\int_0^{\infty} x^k J_{r-1,0,n-r}(x)\, dx = (r-1)!\,(n-r)!\, \nu_{r:n}^{(k)}$$

We immediately obtain that

$$\frac{1}{(r-1)!(n-r)!} \int_0^{\infty} x^k I_{r-1,n-r}(x)\, dx$$

$$= \sum_{1=0}^{r-1} \sum_{1 \le i_1 < \dots i_1 \le n} \nu_{r-1:n-1}^{(k)[i_1, \dots, i_1]}$$

In the same fashion we get

$$\frac{1}{(r-1)!(n-r)!} \int_0^{\infty} x^k I_{n-r,r-1}(x)\, dx$$

$$= \sum_{1=r}^{n} \sum_{1 \le i_1 < \dots i_{n-1} \le n} \nu_{1-r+1:1}^{(k)[i_1, \dots, i_{n-1}]}$$

Now using these expression on the RHS of (5.3.3), we derive the required result.

REMARK 5.3.1: If we set $F_1 = F_2 = \dots = F_n = F$ and $f_1 = f_2 = \dots = f_n = f$ then Relation (5.3.1) reduces to

$$\mu^{(k)}_{r:n} = 2^{-n} \left[\sum_{l=0}^{r-1} \binom{n}{l} \nu^{(k)}_{r-l:n-l} + (-1)^k \sum_{l=r}^{n} \binom{n}{l} \nu^{(k)}_{l-r+1:l} \right]$$

This relation has been devloped by Govindarajulu (1963a).

REMARK 5.3.2: If we set $F_1 = F_2 = \dots = F_n = F$ and $f_1 = f_2 = \dots = f_n = f$ then for single outlier model relation (5.3.1) reduces to

$$\mu^{(k)}_{r:n} = 2^{-n} \left[\sum_{l=0}^{r-1} \binom{n-1}{l} \nu^{(k)}_{r-l:n-l} + (-1)^k \sum_{l=r}^{n} \binom{n-1}{l-1} \nu^{(k)}_{l-r+1:l} \right.$$

$$\left. + \sum_{l=1}^{r-1} \binom{n-1}{l-1} \nu^{*(k)}_{l-r:n-l} + (-1) \sum_{l=r}^{n-1} \binom{n-1}{l} \nu^{*(k)}_{l-r+1:l} \right]$$

This relation has been established recently by Balakrishnan (1988b) and has been used by Balakrishnan and Ambagaspitiya (1988) in studying the robustness properties of various estimators of the location and scale parameters of the double exponential distribution in the presence of single outlier where $\nu^{*(k)}$ denotes the k^{th} moment in the non-outlier case.

RELATION 5.3.2: (Balakrishnan, 1989b)

For $1 \le r < s \le n$,

$$\mu_{r,s:n} = 2^{-n} \left[\sum_{l=0}^{r-1} \sum_{1 \le i_1 < \dots i_l \le n} \nu^{(k)[i_1, \dots, i_l]}_{r-l,s-l:n-l} \right.$$

$$+\sum_{l=s}^{n}\ \sum_{1\le i_1<\dots i_{n-1}\le n} \nu^{(k)[i_1,\dots,i_{n-1}]}_{l-s+1,l-r+1:l}$$

$$-\sum_{l=r}^{s-1}\ \sum_{1\le i_1<\dots i_l\le n} \nu^{[i_1,\dots,i_l]}_{s-l:n-l}\ \nu^{[i_1,\dots,i_n]}_{l-r+1:l}\Bigg] \quad \dots(5.3.4)$$

PROOF: It's proof is identical with previous relation.

REMARK 5.3.3: If we set $F_1 = F_2 = \dots = F_n = F$ and $f_1 = f_2 = \dots = f_n = f$ then Relation (5.3.2) reduces to the corresponding result for the product moments that has been derived by Govindarajulu (1963a).

REMARK 5.3.4: If we set $F_1 = F_2 = \dots = F_{n-1} = F$ and $f_1 = f_2 = \dots = f_{n-1} = f$ for single outlier model, then Relation (5.3.2) reduces to the corresponding result for the product moment that has been applied by Balakrishnan and Ambagaspitiya (1988) in robustness studies.

5.4 PROBABLISTIC PROOF FOR INDEPENDENT NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

Govindarajulu *et al.*(1993) has defined X_i, $i = 1, 2, \dots, n$ are independent random variables with cdf. $F_i(x)$ and pdf. $f_i(x)$, $i = 1, 2, \dots, n$ each symmetric about zero. Let $Y_i = |X_i|$, $i = 1, 2, \dots, n$ and $X_{1:n} \le X_{2:n} \le \dots \le X_{n:n}$ and $Y_{1:n} \le Y_{2:n} \le \dots \le Y_{n:n}$ be the corresponding order statistics. Other notations are same as in section 5.3..

Suppose $X_{r:n} > 0$, then the number of X's ≤ 0 is at the most $r-1$; Let us suppose $X_{i_1}, X_{i_2}, \dots, X_{i_l}$ are only X's ≤ 0. It is then readily seen that conditional distribution of $X_{r:n}$ given that $X_{i_1}, X_{i_2}, \ldots, X_{i_l}$ are negative is same as the unconditional distribution of $Y_{r-l:n-l}^{[i_1,\ldots,i_l]}$. Suppose $X_{r:n} \le 0$ then the number of X's ≤ 0 is at least r. Now by using similar argument, it is seen that the conditional distribution of $X_{r:n}$ given that $X_{i_{n-l+1}}, \ldots, X_{i_n}$ are negative is same as the unconditional distribution of $Y_{l-r+1}^{[i_1,\ldots,i_{n-l}]}$. then the Relation (5.3.1) follows. Similarly Relation (5.3.2) can be followed.

Now assume all X_i 's are not symmetric . Let $\bar{\nu}_{s:n-m}^{[i_1,\ldots,i_m](k)}$ and $\nu_{s,t:n-m}^{[i_1,\ldots,i_m])}$ denote the single and the product moments of order statistics from the conditional distribution of n-m random variables obtained by deleting $X_{i_1}, \ldots, X_{i_m}$ from $X_1, X_2, \ldots, X_n$ given that all these n-m variables are positive. Similarly, let $\bar{\nu}_{s:n-m}^{[i_1,\ldots,i_m](k)}$ and $\bar{\nu}_{s,t:n-m}^{[i_1,\ldots,i_m]}$ denote the corresponding moments of order statistics from the conditional distribution given that all the n-m variables are negative. Now we present the following generalized Relations analogous to the Govindarajulu *et al.*(1993), section 2.3. & 2.4..

For $1 \le r \le n$ and $k = 1, 2, \ldots,$

$$\mu_{r:n}^{(k)} = \sum_{l=0}^{r-1} \sum_{1 \le i_1 < \ldots i_l \le n} \Pi_{(i_1,\ldots,i_l)} \nu_{r-l:n-l}^{[i_1,\ldots,i_l](k)}$$

$$+ \sum_{l=s}^{r-1} \sum_{1 \le i_{l+1} < \ldots i_n \le n} \Pi_{(i_1,\ldots,i_l)} \bar{\nu}_{r:l}^{[i_{l+1},\ldots,i_n](k)} \qquad \ldots(5.4.1)$$

and for $1 \le r < s \le n$,

$$\mu_{r,s:n} = \sum_{l=0}^{r-1} \sum_{1 \le i_1 < \ldots i_l \le n} \Pi_{(i_1,\ldots,i_l)} \nu_{r-1,s-1:n-1}^{[i_1,\ldots,i_l]}$$

$$+ \sum_{l=s}^{r-1} \sum_{1 \le i_{l+1} < \ldots i_n \le n} \Pi_{(i_1,\ldots,i_l)} \bar{\nu}_{r,s:i}^{[i_{l+1},\ldots,i_n]}$$

$$+ \sum_{l=r}^{s-1} \sum_{1 \le i_1 < \ldots i_l \le n} \Pi_{(i_1,\ldots,i_l)} \nu_{s-1:n-1}^{[i_1,\ldots,i_l]} \bar{\nu}_{r:l}^{[i_{l+1}\ldots i_n]} \qquad \ldots(5.4.2)$$

where

$$\Pi_{(i_1,\ldots,i_l)} = P_{i_1} \ldots P_{i_l} Q_{i_{l+1}} \ldots Q_{i_n} \qquad \ldots(5.4.3)$$

with $P_l = P(X_l \le 0) = 1 - Q_l$

REMARK 5.4.1: It is easy to see that the Relations (5.4.1) and (5.4.2) simply reduce to Relations (5.3.1) and (5.3.2) for the special case when all the X's are symmetric about zero. In this case

$$\Pi_{(i_1,\ldots,i_l)} = 2^{-n} \;\forall\; \{i_1\ldots i_l\} \subseteq \{1, 2, \ldots, n\} \;\forall\; l = 0,1,\ldots n$$

5.5 PROBABLISTIC PROOF FOR NON-INDEPENDENT NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

Govindarajulu *et al.*(1993) redefine $\Pi_{(i_1,\ldots,i_l)}$ as

$$\Pi_{(i_1, \ldots, i_l)} = P\left\{ X_{i_1} \leq 0 , \ldots X_{i_l} > 0, X_{i_{l+1}} > 0 \ldots X_{i_n} > 0 \right\}$$

...(5.5.1)

then the Relations (5.4.1) and (5.4.2) continue to hold even for the NI.NI.D. case. It is the case when X_i's jointly have an arbitrary continuous multivariate distribution.

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